Extending Families of Disjoint Hypercyclic Operators

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IAS

8.5.2015
Dynamics of Linear Operators

Let $T$ be a continuous linear operator on a topological vector space $X$. If there is a vector $f \in X$ such that

- $\text{span}\{T^n f : n \geq 1\}$ is dense in $X$, then $T$ is called **cyclic**,
Dynamics of Linear Operators

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- $\text{span}\{T^n f : n \geq 1\}$ is dense in $X$, then $T$ is called **cyclic**, 
- $\{\lambda T^n f : n \geq 1, \lambda \in \mathbb{C}\}$ is dense in $X$, then $T$ is called **supercyclic**.
Dynamics of Linear Operators

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1. $\text{span}\{T^n f : n \geq 1\}$ is dense in $X$, then $T$ is called **cyclic**,
2. $\{\lambda T^n f : n \geq 1, \lambda \in \mathbb{C}\}$ is dense in $X$, then $T$ is called **supercyclic**.
3. $\{T^n f : n \geq 1\}$ is dense in $X$, then $T$ is called **hypercyclic**.
Let $T$ be a continuous linear operator on a topological vector space $X$. If there is a vector $f \in X$ such that

- \( \text{span}\{ T^n f : n \geq 1 \} \) is dense in $X$, then $T$ is called \textbf{cyclic},
- \( \{ \lambda T^n f : n \geq 1, \lambda \in \mathbb{C} \} \) is dense in $X$, then $T$ is called \textbf{supercyclic}.
- \( \{ T^n f : n \geq 1 \} \) is dense in $X$, then $T$ is called \textbf{hypercyclic}.

Such a vector $f$ is said to be a cyclic, supercyclic, or hypercyclic vector for $T$, respectively.
How to Show Hypercyclicity

- $T \in L(X)$ is **topologically transitive** if for any non-empty open subsets $U$ and $V$ of $X$ there exists $n \in \mathbb{N}$ such that

  $$U \cap T^{-n}(V) \neq \emptyset.$$
How to Show Hypercyclicity

- If $X$ is a separable Fréchet space, then $T \in L(X)$ is hypercyclic if and only if it is topologically transitive.

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$$U \cap T^{-n}(V) \neq \emptyset.$$ 

$T \in L(X)$ is **mixing** if for any non-empty open subsets $U$ and $V$ of $X$ there exists $n_0 \in \mathbb{N}$ such that

$$U \cap T^{-n}(V) \neq \emptyset$$

clearly for all $n \geq n_0$. 

How to Show Hypercyclicity

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  for all $n \geq n_0$.

- If $X$ is a separable Fréchet space, then $T \in L(X)$ is hypercyclic if and only if it is topologically transitive.
How to Show Hypercyclicity

\( \mathbf{T} \in L(X) \) is said to satisfy the \textbf{Hypercyclicity Criterion} if there exists dense subsets \( Y, Z \) of \( X \), a strictly increasing subsequence \( (n_k) \subseteq \mathbb{N}^\mathbb{N} \), and maps \( S_k : Z \to X \) such that

1. \( T^{n_k} y \to 0 \) for all \( y \in Y \),
2. \( S_k z \to 0 \) for all \( z \in Z \), and
3. \( T^{n_k} S_k z \to z \) for all \( z \in Z \)

as \( k \to \infty \).
How to Show Hypercyclicity

- \( T \in L(X) \) is said to satisfy the **Hypercyclicity Criterion** if there exists dense subsets \( Y, Z \) of \( X \), a strictly increasing subsequence \( (n_k) \subset \mathbb{N}^\mathbb{N} \), and maps \( S_k : Z \rightarrow X \) such that
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  as \( k \rightarrow \infty \)

- If \( T \) satisfies the Hypercyclicity Criterion, then it is hypercyclic.
Hypercyclicity Criterion

- (Godefroy and Shapiro, 1987) \( T \in L(X) \) satisfies the Hypercyclicity Criterion if and only if it satisfies the **blow up/collapse property**, that is for any \( U, V, \) and \( W \) of non-empty open subsets of \( X \) with \( 0 \in W \) there is some \( n \in N \) with

\[
T^n(U) \cap W \neq \emptyset \text{ and } T^n(W) \cap V \neq \emptyset.
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Hypercyclicity Criterion

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- **(Bès and Peris, 1999)** \(T \in L(X)\) satisfies HC if and only if \(T \oplus T\) is hypercyclic on \(X^2\) (that is \(T\) is **weakly mixing**).
Introduction
Disjoint Hypercyclic Operators
Extending d-Hypercyclic Families

Linear Dynamics
How to Show Hypercyclicity
Weighted Backward Shifts

Hypercyclicity Criterion

- (Godefroy and Shapiro, 1987) $T \in L(X)$ satisfies the Hypercyclicity Criterion if and only if it satisfies the **blow up/collapse property**, that is for any $U$, $V$, and $W$ of non-empty open subsets of $X$ with $0 \in W$ there is some $n \in \mathbb{N}$ with

  $$T^n(U) \cap W \neq \emptyset \text{ and } T^n(W) \cap V \neq \emptyset.$$ 

- (Bès and Peris, 1999) $T \in L(X)$ satisfies HC if and only if $T \oplus T$ is hypercyclic on $X^2$ (that is $T$ is **weakly mixing**).

- (De la Rosa and Reed, 2009) There exists a hypercyclic $T \in L(\ell^1)$ such that $T \oplus T$ is not hypercyclic. Therefore, $T$ fail to satisfy the Hypercyclicity Criterion.
Let \( \{e_n : n \geq 1\} \) denote the canonical basis for \( \ell^2 \).
Unilateral Weighted Backward Shifts

- Let \( \{e_n : n \geq 1\} \) denote the canonical basis for \( \ell^2 \).
- Define \( B_w : \ell^2 \to \ell^2 \) associated with the weight sequence \( w = (w_1, w_2, \ldots) \) by \( B_w e_n = w_n e_{n-1} \) for \( n \geq 2 \) and \( B_w e_1 = 0 \). Thus,
  \[
  B_w(a_1, a_2, a_3, \ldots) = (w_2 a_2, w_3 a_3, \ldots).
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B_w(a_1, a_2, a_3, \ldots) = (w_2 a_2, w_3 a_3, \ldots).
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B_w(a_1, a_2, a_3, \ldots) = (w_2 a_2, w_3 a_3, \ldots).
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- \((\text{Hilden and Wallen, 1974})\) \( B_w \) is always supercyclic.
- \((\text{Salas, 1995})\) \( B_w \) is hypercyclic iff \( \sup_{n \geq 1} \prod_{j=1}^{n} w_j = \infty \).
Let \( \{e_n : n \in \mathbb{Z}\} \) denote the canonical basis for \( \ell^2(\mathbb{Z}) := \{(a_n)_{n=-\infty}^{\infty} : \sum_{n=-\infty}^{\infty} |a_n|^2 < \infty\} \).
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Define \( B_w : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \) associated with the weight sequence \( w = (w_1, w_2, \ldots) \) by \( B_w e_n = w_n e_{n-1} \) for \( n \in \mathbb{Z} \).

(Salas, 1995) \( B_w \) is hypercyclic iff for every \( \epsilon > 0 \) and \( q \in \mathbb{N} \) there exists arbitrarily large \( n \in \mathbb{N} \) such that for every \( i \in \mathbb{Z} \) with \( |i| \leq q \) we have

\[
\prod_{j=0}^{n-1} w_{i+j} > \frac{1}{\epsilon} \quad \text{and} \quad \prod_{j=1}^{n} w_{i-j} < \epsilon.
\]
Bilateral Weighted Backward Shifts

**(Salas, 1999)** $B_w : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is supercyclic iff there exists $(n_k) \subset \mathbb{N}^\mathbb{N}$ s.t. for each $q \in \mathbb{N}$, we have

$$\max \left\{ \left| \frac{\prod_{j=0}^{n_k-1} w_i - j}{\prod_{j=1}^{n_k} w_{\ell+j}} \right| : |i| \leq q \text{ and } |\ell| \leq q \right\} \to 0 \text{ as } k \to \infty,$$
Bilateral Weighted Backward Shifts

- **(Salas, 1999)** $B_w : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is supercyclic iff there exists $(n_k) \subset \mathbb{N}^\mathbb{N}$ s.t. for each $q \in \mathbb{N}$, we have

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- **(Feldman, 2002)** Invertible $B_w : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is hypercyclic iff there exists a strictly increasing sequence $(n_k) \subset \mathbb{N}^\mathbb{N}$ such that

$$\lim_{k \to \infty} \prod_{j=1}^{n_k} w_j = \infty \text{ and } \lim_{k \to \infty} \prod_{j=1}^{n_k} w_{-j} = 0.$$
We say that hypercyclic operators $T_1, \ldots, T_N$ ($N \geq 2$) are **d-hypercyclic** provided that the direct sum operator $T_1 \oplus \ldots \oplus T_N$ acting on $X^N$ have a hypercyclic vector in the form $(f, \ldots, f) \in X^N$. 

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That is the set

$$\{(f, \ldots, f), (T_1f, \ldots, T_Nf), (T_1^2f, \ldots, T_N^2f), \ldots\}$$

is dense in $X^N$. 

"Ozg"ur Martin Extending Families of Disjoint Hypercyclic Operators
Definition

$T_1, \ldots, T_N \in L(X)$ ($N \geq 2$) are **d-topologically transitive** (resp. **d-mixing**) if for all non-empty open $U_0, U_1, \ldots, U_N \subset X$ there exists $n \in \mathbb{N}$ such that (resp. for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$)

$$U_0 \cap T_1^{-n}(U_1) \cap \cdots \cap T_N^{-n}(U_N) \neq \emptyset.$$
Definition

$T_1, \ldots, T_N \in L(X)$ ($N \geq 2$) are \textbf{d-topologically transitive} (resp. \textbf{d-mixing}) if for all non-empty open $U_0, U_1, \ldots, U_N \subset X$ there exists $n \in \mathbb{N}$ such that (resp. for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$)

$$U_0 \cap T_1^{-n}(U_1) \cap \cdots \cap T_N^{-n}(U_N) \neq \emptyset.$$

$T_1, \ldots, T_N \in L(X)$ are d-topologically transitive if and only if they are \textbf{densely} d-hypercyclic.
Theorem (d-Blow-Up/Collapse Property)

Let $T_1, \ldots, T_N$ be $N \geq 2$ operators on $L(X)$. Suppose that for each open neighborhood of zero $W$ of $X$ and non-empty open subsets $V_0, V_1, \ldots, V_N \subset X$ there exists $n \in \mathbb{N}$ so that

$$W \cap T_1^{-n}(V_1) \cap \cdots \cap T_N^{-n}(V_N) \neq \emptyset$$

$$V_0 \cap T_1^{-n}(W) \cap \cdots \cap T_N^{-n}(W) \neq \emptyset$$

Then $T_1, \ldots, T_N$ are d-topologically transitive.
Bès and Peris

Theorem (d-Hypercyclicity Criterion)

\( T_1, \ldots, T_N \in \mathcal{L}(X) \) are d-hypercyclic if there exist dense subsets \( X_0, X_1, \ldots, X_N \) of \( X \), a sequence \( (n_k) \subset \mathbb{N}^\mathbb{N} \), and mappings \( S_{m,k} : X_m \to X \) \((1 \leq m \leq N, \ k \in \mathbb{N})\) satisfying

\[
\begin{align*}
T_{n_k}^m & \xrightarrow[k \to \infty]{} 0 \quad \text{pointwise on } X_0, \\
S_{m,k} & \xrightarrow[k \to \infty]{} 0 \quad \text{pointwise on } X_m, \text{ and} \\
(T_{n_k}^m S_{i,k} - \delta_{i,m} \text{Id}_{X_m}) & \xrightarrow[k \to \infty]{} 0 \quad \text{pointwise on } X_m \ (1 \leq i \leq N).
\end{align*}
\]
Bès and Peris

Theorem

Let $T_m \in L(X)$ ($1 \leq m \leq N$), where $N \geq 2$. The following are equivalent:

(a) $T_1, T_2, \ldots, T_N$ satisfy the $d$-Hypercyclicity Criterion.

(b) For each $r \in \mathbb{N}$, $\underbrace{T_1 \oplus \cdots \oplus T_1}_r, \ldots, \underbrace{T_N \oplus \cdots \oplus T_N}_r$ are $d$-topologically transitive operators in $L(X^r)$.
Theorem

Let $T_m \in L(X)$ $(1 \leq m \leq N)$, where $N \geq 2$. The following are equivalent:

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(b) For each $r \in \mathbb{N}$, $T_1^r \oplus T_1^r, \ldots, T_N^r \oplus T_N^r$ are d-topologically transitive operators in $L(X^r)$.

Definition

$T_1, T_2 \in X$ are called **d-weakly mixing** if $T_1 \oplus T_1, T_2 \oplus T_2$ are d-hypercyclic on $X^2$. 

Bès and Peris
Powers of Weighted Shifts

Theorem (Bès and Peris, 2007)

Let $B_1, \ldots, B_N$ ($N \geq 2$) be unilateral (or bilateral) weighted shifts and $1 \leq r_1 < \cdots < r_N$. Then $B_1^{r_1}, \ldots, B_N^{r_N}$ are d-hypercyclic if and only if they satisfy the d-Hypercyclicity Criterion.
d-Hypercyclic Unilateral Shifts

Theorem (Bès, M, and Sanders, 2014)

Let $B_1, B_2$ be unilateral weighted shifts on $\ell^2$ with weights $(w_j^{(1)})$ and $(w_j^{(2)})$ and define $\alpha_{i,n} := \prod_{j=1}^{n} w_{i+j}/w_{i+j}$. TFAE:

(a) $B_1, B_2$ are $d$-hypercyclic.

(b) $B_1, B_2$ are densely $d$-hypercyclic and they satisfy the $d$-blow up/collapse property.

(c) There exists $(n_k) \subset \mathbb{N}^\mathbb{N}$ s.t. for each $i \in \mathbb{N}$, we have

$$\left| \prod_{j=1}^{n_k} w_{i+j}^{(1)} \right| \to \infty \text{ as } k \to \infty$$

and the set

$$\{(\alpha_0,n_k, \alpha_1,n_k, \alpha_2,n_k, \ldots) : k \geq 1\}$$

is dense in $\mathbb{C}^\mathbb{N}$ w.r.t. the product topology.
d-Hypercyclic Bilateral Shifts

**Theorem (Bès, M, and Sanders, 2014)**

Let $B_1, B_2$ be bilateral weighted shifts on $\ell^2(\mathbb{Z})$ with weights

\[
\{w_j^{(1)} : j \in \mathbb{Z}\} \text{ and } \{w_j^{(2)} : j \in \mathbb{Z}\}.
\]

TFAE:

(a) $B_1, B_2$ are $d$-hypercyclic.

(b) $B_1, B_2$ are densely $d$-hypercyclic and they satisfy the $d$-blow up/collapse property.

(c) There exists $(n_k) \subset \mathbb{N}^\mathbb{N}$ s.t. for each $i \in \mathbb{N}$, we have

\[
\left| \prod_{j=1}^{n_k} w_{i+j}^{(1)} \right| \to \infty \text{ and } \left| \prod_{j=0}^{n_k-1} w_{i-j}^{(m)} \right| \to \text{ as } k \to \infty \text{ (} m = 1, 2 \text{)}
\]

and the set

\[
\left\{ (\ldots, \alpha_{-2}, n_k, \alpha_{-1}, n_k, \alpha_0, n_k, \alpha_1, n_k, \alpha_2, n_k, \ldots) : k \geq 1 \right\}
\]

is dense in $\mathbb{C}^\mathbb{Z}$ w.r.t. the product topology.
Results

- **Theorem (Bès, M, and Sanders, 2014)**

Let $B_1, \ldots, B_N$ ($N \geq 2$) be unilateral or bilateral weighted shifts. Then they never satisfy the **d-Hypercyclicity Criterion** although they always satisfy the **Blow Up/Collapse Proposition** if they are $d$-hypercyclic.
Results

- Theorem (Bès, M, and Sanders, 2014)

Let $B_1, \ldots, B_N$ $(N \geq 2)$ be unilateral or bilateral weighted shifts. Then they never satisfy the d-Hypercyclicity Criterion although they always satisfy the Blow Up/Collapse Proposition if they are d-hypercyclic.

- Theorem (Sanders and Shkarin, 2014)

Being d-weakly mixing doesn’t imply satisfying the d-Hypercyclicity Criterion.
Introduction

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Recent Results

Results

- Theorem (Bès, M, and Sanders, 2014)

Let $B_1, \ldots, B_N$ ($N \geq 2$) be unilateral or bilateral weighted shifts. Then they never satisfy the d-Hypercyclicity Criterion although they always satisfy the Blow Up/Collapse Proposition if they are d-hypercyclic.

- Theorem (Sanders and Shkarin, 2014)

Being d-weakly mixing doesn’t imply satisfying the d-Hypercyclicity Criterion.

- Theorem (Sanders and Shkarin, 2014)

Being d-hypercyclic doesn’t imply being densely d-hypercyclic (i.e. being d-topologically transitive).
Summary

\[
\text{d-mixing} \
\Downarrow \\
\text{d-Hypercyclicity Criterion} \
\Downarrow \\
\text{d-weakly mixing} \
\Downarrow \\
\text{d-blow-up/collapse} \
\Downarrow \\
\text{d-topologically transitive} \Leftrightarrow \text{densely d-hypercyclic} \
\Downarrow \\
d\text{-hypercyclic}
\]
Problem (Salas, 2013)

Given a finite collection $T_1, \ldots, T_N$ of $d$-hypercyclic (d-supercyclic) operators, can one find an additional operator $T_{N+1}$ for which the larger family $T_1, \ldots, T_N, T_{N+1}$ remains hypercyclic (d-supercyclic)?
A Partial Answer

- **Theorem (Shkarin, 2010)**

  If $T_1 \in L(X)$ satisfy the Hypercyclicity Criterion, then there exists $T_2 \in L(X)$ such that $T_1, T_2$ are $d$-hypercyclic.
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**Theorem (Shkarin, 2010)**

*If* \( T_1 \in L(X) \) *satisfy the Hypercyclicity Criterion, then there exists* \( T_2 \in L(X) \) *such that* \( T_1, T_2 \) *are d-hypercyclic.*

**Proof:** Then \( T_1 \oplus T_1 \) is hypercyclic on \( X^2 \).
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- **Theorem (Shkarin, 2010)**

  If $T_1 \in L(X)$ satisfy the Hypercyclicity Criterion, then there exists $T_2 \in L(X)$ such that $T_1, T_2$ are d-hypercyclic.

  - **Proof:** Then $T_1 \oplus T_1$ is hypercyclic on $X^2$.
  - Let $(f, g)$ be a hypercyclic vector for $T_1 \oplus T_1$. 
Theorem (Shkarin, 2010)

If $T_1 \in L(X)$ satisfy the Hypercyclicity Criterion, then there exists $T_2 \in L(X)$ such that $T_1, T_2$ are d-hypercyclic.

Proof: Then $T_1 \oplus T_1$ is hypercyclic on $X^2$.

Let $(f, g)$ be a hypercyclic vector for $T_1 \oplus T_1$.

We can find an invertible $S \in L(X)$ such that $Sg = f$. 
A Partial Answer

▶ Theorem (Shkarin, 2010)

If $T_1 \in L(X)$ satisfy the Hypercyclicity Criterion, then there exists $T_2 \in L(X)$ such that $T_1, T_2$ are $d$-hypercyclic.

▶ Proof: Then $T_1 \oplus T_1$ is hypercyclic on $X^2$.
▶ Let $(f, g)$ be a hypercyclic vector for $T_1 \oplus T_1$.
▶ We can find an invertible $S \in L(X)$ such that $Sg = f$.
▶ Then $(f, f)$ is a hypercyclic vector for $T_1 \oplus S^{-1}T_1S$. 
A Partial Answer

- **Theorem (Shkarin, 2010)**

  *If* \( T_1 \in L(X) \) *satisfy the Hypercyclicity Criterion, then there exists* \( T_2 \in L(X) \) *such that* \( T_1, T_2 \) *are d-hypercyclic.*

  - **Proof:** Then \( T_1 \oplus T_1 \) is hypercyclic on \( X^2 \).
  - Let \( (f, g) \) be a hypercyclic vector for \( T_1 \oplus T_1 \).
  - We can find an invertible \( S \in L(X) \) such that \( Sg = f \).
  - Then \( (f, f) \) is a hypercyclic vector for \( T_1 \oplus S^{-1} T_1 S \).
  - Let \( T_2 = S^{-1} T_1 S \). \( \square \)
Questions

▶ **Question 1**: What about d-hypercyclic $T_1, T_2$?
Questions

• **Question 1:** What about d-hypercyclic $T_1, T_2$?

• **Question 2:** If $T_1$ is a weighted shift, can we choose $T_2$ as another weighted shift?
Theorem (M and Sanders)

Let $B_1, B_2$ be unilateral weighted shifts on $\ell^2$ with weights $(w_j^{(m)})$ for $m = 1, 2$, and define

$$\alpha_{i,n} := \prod_{j=1}^{n} \frac{w_{i+j}^{(2)}}{w_{i+j}^{(1)}} = \frac{w_{i+1}^{(2)} w_{i+2}^{(2)} \ldots w_{i+n}^{(2)}}{w_{i+1}^{(1)} w_{i+2}^{(1)} \ldots w_{i+n}^{(1)}}$$

TFAE:

(a) $B_1, B_2$ are $d$-supercyclic.
(b) $B_1, B_2$ are densely $d$-supercyclic.
(c) There exists $(n_k) \subset \mathbb{N}^\mathbb{N}$ s.t. the set

$$\{ (\alpha_0, n_k, \alpha_1, n_k, \alpha_2, n_k, \ldots) : k \geq 1 \}$$

is dense in $\mathbb{C}^\mathbb{N}$ w.r.t. the product topology.
Characterization of d-Supercyclic Bilateral Shifts

Theorem (M and Sanders)

Let $B_1, B_2$ be bilateral weighted shifts on $\ell^2(\mathbb{Z})$ with weights $\{w_j^{(1)} : j \in \mathbb{Z}\}$ and $\{w_j^{(2)} : j \in \mathbb{Z}\}$. TFAE:

(a) $B_1, B_2$ are d-supercyclic.

(b) $B_1, B_2$ are densely d-supercyclic.

(c) There exists $(n_k) \subset \mathbb{N}^\mathbb{N}$ s.t. for each $q \in \mathbb{N}$, we have

$$\max \left\{ \left| \prod_{j=0}^{n_k-1} \frac{w_{i-j}^{(2)}}{\prod_{j=1}^{n_k} w_{\ell+j}^{(1)}} \right| : |i| \leq q \text{ and } |\ell| \leq q \right\} \rightarrow 0 \text{ as } k \rightarrow \infty,$$

and the set $\{(\ldots, \alpha_{-2,n_k}, \alpha_{-1,n_k}, \alpha_{0,n_k}, \alpha_{1,n_k}, \alpha_{2,n_k}, \ldots) : k \geq 1 \}$ is dense in $\mathbb{C}^\mathbb{Z}$ w.r.t. the product topology.
Corollary

Let $B_1, B_2$ be weighted shifts on $\ell^2(\mathbb{Z})$ such that $B_1$ is invertible. Then $B_1, B_2$ are not d-supercyclic.
Corollary

Let $B_1, B_2$ be weighted shifts on $\ell^2(\mathbb{Z})$ such that $B_1$ is invertible. Then $B_1, B_2$ are not $d$-supercyclic.

Proof:

It suffices to establish the result for just two bilateral weighted shifts. For integers $1 \leq m \leq 2$, let $\{w_j^{(m)} : j \in \mathbb{Z}\}$ be the weight sequence for $B_m$, and set $\alpha = \inf \{|w_j^{(1)} : j \in \mathbb{Z}\}$ and $\beta = \sup \{|w_j^{(2)} : j \in \mathbb{Z}\}$. Now, by way of contradiction, suppose the shifts $B_1, B_2$ are $d$-hypercyclic.
Thus, there exists a strictly increasing sequence \((m_k)_{k=0}^{\infty}\) of positive integers such that

\[
\prod_{j=1}^{m_k} \frac{w_j^{(2)}}{w_j^{(1)}} \rightarrow w_1^{(2)} \alpha \quad \text{and} \quad \prod_{j=1}^{m_k} \frac{w_{1+j}^{(2)}}{w_{1+j}^{(1)}} \rightarrow 2w_1^{(1)} \beta \quad \text{as} \ k \rightarrow \infty.
\]

Thus,

\[
\frac{|w_1^{(2)}| \alpha}{|w_1^{(1)}| \beta} \leq \left| \frac{w_1^{(2)} w_{m_k+1}^{(1)}}{w_1^{(1)} w_{m_k+1}^{(2)}} \right| = \left| \prod_{j=1}^{m_k} \frac{w_j^{(2)}}{w_j^{(1)}} \right| = \frac{1}{2} \frac{|w_1^{(2)}| \alpha}{|w_1^{(1)}| \beta}
\]

as \(k \rightarrow \infty\).
Corollary (M and Sanders)

There exists $d$-supercyclic weighted shifts $B_1, B_2$ on $\ell^2$ (or on $\ell^2(\mathbb{Z})$) such that $B_1, B_2, B_3$ fail to be $d$-supercyclic for any weighted shift $B_3$. 
Corollary (M and Sanders)

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Theorem (M and Sanders)

If $B_1, B_2, \ldots, B_N$ are d-supercyclic weighted shifts on $\ell^2$ (or $\ell^2(\mathbb{Z})$), then there exists an operator $T_{N+1}$ such that $B_1, B_2, \ldots, B_N, T_{N+1}$ remain d-supercyclic. Moreover, if $f$ is a d-supercyclic vector for the shifts $B_1, B_2, \ldots, B_N$, then the operator $T_{N+1}$ may be chosen such that $f$ remains a d-supercyclic vector for the operators $B_1, B_2, \ldots, B_N, T_{N+1}$. 
Idea of the proof:

- Let $f$ be a $d$-supercyclic vector for $B_1, \ldots, B_N$. 
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- Let $f$ be a $d$-supercyclic vector for $B_1, \ldots, B_N$.
- Find $g$ such that $(f, \ldots, f, g)$ is a supercyclic vector for $B_1 \oplus \cdots \oplus B_N \oplus B_1$. 
Idea of the proof:

- Let $f$ be a $d$-supercyclic vector for $B_1, \ldots, B_N$.
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- Choose an invertible operator $L$ s.t. $Lg = f$. 
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- Let $T_{N+1} = L^{-1}B_1L$. 
Idea of the proof:

- Let $f$ be a $d$-supercyclic vector for $B_1, \ldots, B_N$.
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- Thus $(f, \ldots, f, f)$ is supercyclic for $B_1 \oplus \cdots \oplus B_N \oplus T_{N+1}$.
Idea of the proof:

- Let $f$ be a d-supercyclic vector for $B_1, \ldots, B_N$.
- Find $g$ such that $(f, \ldots, f, g)$ is a supercyclic vector for $B_1 \oplus \cdots \oplus B_N \oplus B_1$.
- Choose an invertible operator $L$ s.t. $Lg = f$.
- Let $T_{N+1} = L^{-1}B_1L$.
- Thus $(f, \ldots, f, f)$ is supercyclic for $B_1 \oplus \cdots \oplus B_N \oplus T_{N+1}$.
- This means $f$ is a d-supercyclic vector for $B_1, \ldots, B_N, T_{N+1}$.
How to find \( g \):

1. Let \( \{(x_r^{(1)}, \ldots, x_r^{(N)}, y_r) : r \geq 1\} \) be dense in \( (l^2)^{N+1} \)
   such that \( x_r^{(1)}, \ldots, x_r^{(N)}, y_r \in \text{span}\{e_1, \ldots, e_r\} \).
How to find $g$:

- Let $\{(x_r^{(1)}, \ldots, x_r^{(N)}, y_r) : r \geq 1\}$ be dense in $(\ell^2)^{N+1}$ such that $x_r^{(1)}, \ldots, x_r^{(N)}, y_r \in \text{span}\{e_1, \ldots, e_r\}$.

- Choose $\lambda_r \in \mathbb{C}$ and $n_r \in \mathbb{N}$ such that $\|\lambda_r B_{n_r} f - x_r^{(m)}\| \to 0$ as $r \to \infty$ for $1 \leq m \leq N$. 

\[ g := \sum_{r=1}^{\infty} \lambda_r \sum_{n=1}^{n_r} S_n y_r \] 

Also such that $g$ is in $\ell^2$ where $S_n := \frac{1}{w_n + 1} e_{n+1}$ and $(w_n)$ is the weight sequence of $B_1$. 

Also such that as $r \to \infty$ we have $\|\lambda_r B_{n_r} f - y_r\| \leq \|r - 1 \sum_{t=1}^{r-1} \lambda_t \lambda_{r-t} B_{n_r} f - n_t y_t\| + |\lambda_r| \|B_{n_r} f\| \to 0$. 

\[ \sum_{t=r+1}^{\infty} \|1_{\lambda_t} S_{n_t} y_t\| \to 0. \]
How to find $g$:

- Let $\{(x_r^{(1)}, \ldots, x_r^{(N)}, y_r) : r \geq 1\}$ be dense in $(\ell^2)^{N+1}$ such that $x_r^{(1)}, \ldots, x_r^{(N)}, y_r \in \text{span}\{e_1, \ldots, e_r\}$.

- Choose $\lambda_r \in \mathbb{C}$ and $n_r \in \mathbb{N}$ such that $\|\lambda_r B_{n_r} f - x_r^{(m)}\| \to 0$ as $r \to \infty$ for $1 \leq m \leq N$.

- Also such that $g := \sum_{r=1}^{\infty} \frac{1}{\lambda_r} S^{n_r} y_r$ is in $\ell^2$ where $S_{e_n} := \frac{1}{w_{n+1}} e_{n+1}$ and $(w_j)$ is the weight sequence of $B_1$. 
How to find $g$:

- Let $\{(x_r^{(1)}, \ldots, x_r^{(N)}, y_r) : r \geq 1\}$ be dense in $(\ell^2)^{N+1}$ such that $x_r^{(1)}, \ldots, x_r^{(N)}, y_r \in \text{span}\{e_1, \ldots, e_r\}$.

- Choose $\lambda_r \in \mathbb{C}$ and $n_r \in \mathbb{N}$ such that $\|\lambda_r B_{m}^{n_r} f - x_r^{(m)}\| \to 0$ as $r \to \infty$ for $1 \leq m \leq N$.

- Also such that $g := \sum_{r=1}^{\infty} \frac{1}{\lambda_r} S^{n_r} y_r$ is in $\ell^2$ where $S e_n := \frac{1}{w_{n+1}} e_{n+1}$ and $(w_j)$ is the weight sequence of $B_1$.

- Also such that as $r \to \infty$ we have

$$\|\lambda_r B_1^{n_r} g - y_r\| \leq \left\| \sum_{t=1}^{r-1} \frac{\lambda_r}{\lambda_t} B_1^{n_r-n_t} y_t \right\| + \|\lambda_r\| B_1^{n_r} \left\| \sum_{t=r+1}^{\infty} \frac{1}{\lambda_t} S^{n_t} y_t \right\|$$

which goes to zero.
Thanks

Thank you all for attending.