SEMIGROUP APPROACH FOR PARTIAL DIFFERENTIAL EQUATIONS OF EVOLUTION

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Evolution Equations

\[ u(x, t) \]

(i) Does it exist?
(ii) Is it unique?
(iii) Does it depend continuously on the initial data?
⇒ Well-posed?

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⇒ Well-posed?
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Weak solution!!
Solutions

**Classical solution**: A solution of a "PDE of order $k$" which is at least $k$ times continuously differentiable.

**Weak solution!!**

**Sobolev space**: Hilbert space which involves weak solutions $H^k(\mathbb{R})$. 
Simple example

\[
\frac{du}{dt} = f(u), \quad u(0) = u_0 \tag{1}
\]

Two Banach spaces: \(X\) and \(Y\), with \(Y \subset X\).
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Assume that for all \( u_0 \), there exist a real number \( T > 0 \) and a unique solution \( u \in C([0, T], Y) \) satisfying equation (1).
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\[ \Rightarrow \frac{du}{dt} \in C([0, T], X)). \]

Assume that \( u_0 \mapsto u \) is continuous (\( Y \mapsto C([0, T], Y) \).

\[ \Rightarrow \text{locally well-posedness in } Y. \]

Very strong!!
Homogeneous equation:

\[ u'(t) + Au(t) = 0, \quad 0 \leq t \leq T, \]
\[ u(0) = u_0 \]
∀ \ u_0, \ solution \ u \ exists \ and \ is \ unique:

\[ u(t) = e^{tA}u_0 \]

Here, \[ e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \].
Nonhomogeneous equation:

\[ u'(t) + Au(t) = f(t), \quad 0 \leq t \leq T, \]
\[ u(0) = u_0 \]
Linear Evolution Equations

\[ u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}f(s)\,ds \]

Lions, Yosida, Pazy…
KATO! Semigroup theory!!
Definition

Let $X$ be a Banach space. If the mapping $T(t) : \mathbb{R}^+ \rightarrow \mathcal{L}(X)$ satisfies the following properties, then it is a $C_0$ semigroup:

(i) $T(0) = I$

(ii) $\forall t, s \geq 0, T(t+s) = T(t)T(s)$

(iii) $\lim_{t \to 0} T(t)x = x, \ x \in X$. 

Definition

Let \( \{ T(t) \}_{t \geq 0} \) be a \( C_0 \) semigroup. If the limit \( Ax := \lim_{t \to 0} \frac{T(t)x - x}{t} \) is defined, then the infinitesimal generator of \( T(t) \) is \( A \). Domain of \( A \) is denoted by \( D(A) \):

\[
D(A) := \{ x \in X : \lim_{t \to 0} \frac{T(t)x - x}{t} \text{ limit is in } X \}.
\]
Cauchy Problem

\[ u'(t) = Au(t), \quad u(0) = u_0, \]
Cauchy Problem

\[ u'(t) = Au(t), \quad u(0) = u_0, \]

(i) If a continuously differentiable function \( u \) satisfies \( u(t) \in D(A) \) for all \( t \geq 0 \) and satisfies the equation, then it is a classical solution,
Cauchy Problem

\[ u'(t) = Au(t), \quad u(0) = u_0, \]

(i) If a continuously differentiable function \( u \) satisfies \( u(t) \in D(A) \) for all \( t \geq 0 \) and satisfies the equation, then it is a classical solution,

(ii) If for a continuous function \( u \), \( \int_0^t u(s)ds \in D(A) \) and \( A \int_0^t u(s)ds = u(t) - x \), then it is a mild solution.
Idea

\[ u'(t) + A(u, t)u = 0, \quad u(0) = u_0 \]

\[ \rightarrow u(t) = e^{tA}u_0 \rightarrow u(t) = T(t)u_0 \]
Idea

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\[ u'(t) + A(\omega, t)u = 0, \quad u(0) = u_0 \]

\[ \rightarrow u(t) = T_\omega(t)u_0 \]
Idea

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\[ u'(t) + A(\omega, t)u = 0, \quad u(0) = u_0 \]
\[ \rightarrow u(t) = T_\omega(t)u_0 \]
\[ \omega \mapsto \Phi(\omega) = T_\omega(t)u_0 = u \]
\[ \Phi(\omega) \text{ contraction mapping} \Rightarrow \text{unique solution } u \text{ (Banach fixed point theorem)} \]
Notation

- $B(X; Y)$: Set of all bounded linear operators on $X$ to $Y$
- $G(X)$: All negative generators of $C_0$ semigroups on $X$
- $G(X, \mu, \beta)$: $-A$ generates a $C_0$ semigroup $\{e^{-tA}\}$ with $\|e^{-tA}\| \leq \mu e^{\beta t}, 0 \leq t < \infty$
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$A \in G(X, 1, \beta) \Rightarrow$ quasi-$m$-accretive
Heat equation

\[ u_t - \kappa u_{xx} = 0, \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R} \]
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Solution by classical method:

\[ u(x, t) = \frac{1}{\sqrt{4\pi \kappa t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\kappa t}} u_0(y) dy \]
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By new notation,

\[ u_t + Au = 0, \quad u(0) = u_0 \]

where \( A = -\kappa D_x^2 \) and \( T(t)u_0 = \frac{1}{\sqrt{4\pi\kappa t}} (e^{-\frac{x^2}{4\kappa t}} \ast u_0)(x) \).
Heat equation

\[ u_t - \kappa u_{xx} = 0, \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R} \]

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For all \( t \), \( \|u(t)\| \leq u_0 \Rightarrow -\kappa D_x^2 \in G(L^2, 1, 0) \).
General approach

Evolution equation depending on time variable $t$ and spatial variable $x$
General approach

Evolution equation depending on time variable $t$ and spatial variable $x$

$\rightarrow$ Ordinary differential equation depending on time variable $t$ in an infinite dimensional Banach space
General approach

Evolution equation depending on time variable \( t \) and spatial variable \( x \)

→ Ordinary differential equation depending on time variable \( t \) in an infinite dimensional Banach space

- Choose an appropriate Banach space.
- Choose the corresponding operator \( A \).
- Spatial derivatives and other restrictions are captured by either the Banach space or the operator.
Quasi-linear equations

\[ u_t + A(u)u = f(u), \quad t \geq 0, \quad u(0) = u_0. \quad (2) \]

\( X \) and \( Y \) Banach spaces
\( Y \subset X \) continuous and densely injected
\( S : Y \leftrightarrow X \) topological isomorphism
Assumptions

(A1) For given \( C > 0 \) and for all \( y \in Y \) satisfying \( \|y\|_Y \leq C \), \( A(y) \) is quasi-m-accretive. In other words, \( -A(y) \) generates a quasi-contractive \( C_0 \) semigroup \( \{ T(t) \}_{t \geq 0} \) in \( X \) satisfying \( \| T(t) \| \leq e^{wt} \).
Assumptions

(A1) For given $C > 0$ and for all $y \in Y$ satisfying $\|y\|_Y \leq C$, $A(y)$ is quasi-m-accretive. In other words, $-A(y)$ generates a quasi-contractive $C_0$ semigroup $\{T(t)\}_{t \geq 0}$ in $X$ satisfying $\|T(t)\| \leq e^{wt}$.

(A2) For all $y \in Y$, $A(y)$ is a bounded linear operator from $Y$ to $X$ and

$$\|(A(y) - A(z))w\|_X \leq c_1 \|y - z\|_X \|w\|_Y, \quad y, z, w \in Y.$$
Assumptions

\[(A3)\] \(SA(y)S^{-1} = A(y) + B(y)\), where \(B(y) \in \mathcal{L}(X)\) is uniformly bounded on \(\{y \in Y : \|y\|_Y \leq M\}\) and
\[
\|(B(y) - B(z))w\|_X \leq c_2\|y - z\|_Y\|w\|_X, \quad w \in X.
\]
Assumptions

(A3) $SA(y)S^{-1} = A(y) + B(y)$, where $B(y) \in \mathcal{L}(X)$ is uniformly bounded on $\{y \in Y : \|y\|_Y \leq M\}$ and

$$||(B(y) - B(z))w||_X \leq c_2\|y - z\|_Y\|w\|_X, \quad w \in X.$$ 

(A4) The function $f$ is bounded on bounded subsets $\{y \in Y : \|y\|_Y \leq M\}$ of $Y$ for each $M > 0$, and is Lipschitz in $X$ and $Y$:

$$\|f(y) - f(z)\|_X \leq c_3\|y - z\|_X, \quad \forall y, z \in X$$

and

$$\|f(y) - f(z)\|_Y \leq c_4\|y - z\|_Y, \quad \forall y, z \in Y.$$ 

Here, $c_1, c_2, c_3$ and $c_4$ are constants depending only on $\max\{\|y\|_Y, \|z\|_Y\}$. 
1

Theorem

Assume (A1), (A2), (A3), (A4) hold. Given \( u_0 \in Y \), there is a maximal \( T > 0 \), depending on \( u_0 \) and a unique solution \( u \) to (2) such that

\[
u = (u_0, .) \in C([0, T), Y) \cap C^1([0, T), X)\).
\]

Moreover, the map \( u_0 \mapsto u(u_0, .) \) is continuous from \( Y \) to \( C([0, T), Y) \cap C^1([0, T), X) \).

Quasilinear wave equation

\[ u_t + a(u)u_x = b(u)u \]

- \( X = L^2(\mathbb{R}), \ Y = H^2(\mathbb{R}), \ S = 1 - D_x^2 \)
- \( A(u, t) = A(u) = a(u)\partial_x \)
- \( f(u, t) = b(u)u \)
- \( B(u) = -[a'(u)u_{xx} + a''(u)u_x^2]\partial S^{-1} - 2a'(u)u_x\partial^2 S^{-1} \)
Quasilinear wave equation

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Alternatively,

- \( X = L^2(\mathbb{R}) \), \( Y = H^s(\mathbb{R}) \), \( s \geq 2 \),
  - \( S = (1 - D_x^2)^{s/2} \) or \( S = (1 - D_x)^s \)

Gain regularity!!
Here, $u(x, t)$ denotes free surface elevation, $\varepsilon$ and $\mu$ represent the amplitude and shallowness parameters, respectively.
Rewrite equation (3) in the following way and define the periodic Cauchy problem:

\[ u_t + A(u)u = f(u) \quad x \in \mathbb{R}, \ t > 0, \]
\[ u(x, 0) = u_0(x) \quad x \in \mathbb{R}, \]
\[ u(x, t) = u(x + 1, t) \quad x \in \mathbb{R}, \ t > 0, \]

where

\[ A(u) = -(1 + \frac{7}{2}\varepsilon u)\partial_x, \]
\[ f(u) = -(1 - \frac{\mu}{12}\partial_x^2)^{-1}\partial_x[2u + \frac{5}{2}\varepsilon u^2 - \frac{1}{8}\varepsilon^2 u^3 + \frac{3}{64}\varepsilon^3 u^4 - \frac{7}{48}\varepsilon\mu u_x^2]. \]
Setting

(i) \( X = L^2[0, 1] \),
(ii) \( Y = H^s_{\text{per}}, s > 3/2 \),
(iii) \( S = \Lambda^s, \Lambda = (1 - \partial_x^2)^{1/2} \).
Assumption 1

- If $u \in H^s_{\text{per}}$, $s > 3/2$, then the operator $A(u) = -(1 + u)\partial_x$ with the following domain,

$$\mathcal{D}(A) = \{ \omega \in L^2[0, 1] : -(1 + u)\partial_x \omega \in L^2[0, 1] \} \subset L^2[0, 1]$$

is quasi-m-accretive.
Assumption 1

If

(a) For all \( w \in D(A) \), there exists a real number \( \beta \) such that
\[
(Aw, w)_X \geq -\beta \|w\|_X^2,
\]
(b) The range of \( A + \lambda I \) is all of \( X \) for some (or all) \( \lambda > \beta \),

then we are done.
Assumption 2

- For every $\omega \in H^s_{per}$, $s > 3/2$, $A(u)$ is bounded linear operator from $H^s_{per}$ to $L^2[0,1]$ and

$$
\|(A(u) - A(v))\omega\|_{L^2[0,1]} \leq c_1 \|u - v\|_{L^2[0,1]} \|\omega\|_{H^s_{per}}.
$$
Assumption 3

The operator

\[ B(u) = SA(u)S^{-1} - A(u) = \Lambda^s (u \partial_x + \partial_x)\Lambda^{-s} - (u \partial_x + \partial_x) = [\Lambda^s, u \partial_x + \partial_x]\Lambda^{-s} \]

is bounded in \( L^2[0, 1] \) for \( u \in H^s_{per} \) with \( s > 3/2 \).
Let $f(u)$ be the function (8). Then:

(i) $f$ is bounded on bounded subsets \( \{ u \in H_{per}^s : \| u \|_{H_{per}^s} \leq M \} \) of $H_{per}^s$ for each $M > 0$.

(ii) $\| f(u) - f(v) \|_{L^2[0,1]} \leq c_3 \| u - v \|_{L^2[0,1]}$.

(iii) $\| f(u) - f(v) \|_{H_{per}^s} \leq c_4 \| u - v \|_{H_{per}^s}, s > 3/2$. 
Local well-posedness

Given $u_0 \in H^s_{per}$, $s > \frac{3}{2}$, there exists a unique solution $u$ to (3)-(6) depending on $u_0$ as follows:

$$u = u(u_0, \cdot) \in C([0, T), H^s_{per}) \cap C^1([0, T), L^2[0, 1]).$$

Moreover, the mapping $u_0 \in H^s_{per} \mapsto u(u_0, \cdot)$ is continuous from $H^s_{per}$ to $C([0, T), H^s_{per}) \cap C^1([0, T), L^2[0, 1]).$
