Lu Qi-Keng problem

Istanbul Analysis Seminars

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December 11, 2015
Definition of the Bergman kernel function

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$$f(z) = \int_D f(w)\overline{K_z(w)}dV(w)$$

for all $f \in L^2_a(D)$. 
Let $D$ be a bounded domain in $\mathbb{C}^n$. The Bergman space $L^2_a(D)$ is the space of holomorphic square integrable functions on $D$. For any $z \in D$, $\Phi_z : L^2_a(D) \rightarrow \mathbb{C}$ defined by $\Phi_z(f) = f(z)$ is a linear functional on $L^2(D)$ and by the Riesz representation theorem, there is a unique $K_z(\cdot) \in L^2_a(D)$ such that

$$f(z) = \int_D f(w)\overline{K_z(w)}dV(w)$$

for all $f \in L^2_a(D)$. Define $K_D(z, w) = \overline{K_z(w)}$, the Bergman kernel for $D$. 


Another representation

The Bergman kernel depends on the choice of $D$ and is also represented by

$$K_D(z, w) = \sum_{j=0}^{\infty} \phi_j(z)\overline{\phi_j(w)}, \quad (z, w) \in D \times D,$$

where $\{\phi_j(\cdot): j = 0, 1, 2, \ldots\}$ is a complete orthonormal basis for $L^2_a(D)$. 

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where $\{\phi_j(\cdot): j = 0, 1, 2, \ldots\}$ is a complete orthonormal basis for $L^2_a(D)$. If $D$ is the Hermitian unit ball $B_n$ defined by

$$B_n = \{z \in \mathbb{C}^n: |z_1|^2 + |z_2|^2 + \ldots + |z_n|^2 < 1\},$$

then its Bergman kernel is written in the closed form, so that,

$$K_{B_n}(z, w) = \frac{n!}{\pi^n} \frac{1}{(1 - \langle z, w \rangle)^{n+1}},$$

where $\langle z, w \rangle := z_1\overline{w}_1 + \ldots + z_n\overline{w}_n$. 

Properties of the Bergman kernel function

- The Bergman kernel \( K_D(z, w) \) is conjugate symmetric:
  \[
  K_D(z, w) = 
  \overline{K_D(w, z)}
  \]

- The Bergman kernel is uniquely determined by the properties that it is an element of \( L^2_a(D) \) in \( z \), is conjugate symmetric, and reproduces \( L^2_a(D) \).

- For \( z \in D \subset \subset \mathbb{C}^n \) it holds that \( K_D(z, z) > 0 \). Moreover
  \[
  K_D(z, z) = \sup_{f \in L^2_a(D)} \frac{|f(z)|^2}{\|f\|^2_{L^2_a(D)}} = \sup_{\|f\|_{L^2_a(D)} = 1} |f(z)|^2
  \]
Properties of the Bergman kernel function

- Let $D_1, D_2$ be domains in $\mathbb{C}^n$. Let $F : D_1 \to D_2$ be biholomorphic. Then

  $$K_{D_1}(z, \zeta) = K_{D_2}(F(z), F(\zeta)) J_C(F(z)) J_C(F(\zeta)).$$

- Note also that if $(\phi_j)_j$ and $(\psi_k)_k$ are orthonormal bases of $L^2_a(D_1)$ and $L^2_a(D_2)$ respectively, then $(\phi_j \times \psi_k)_j,k$ forms an orthonormal basis for $L^2_a(D_1 \times D_2)$ (here $(f \times g)(z, w) := f(z)g(w)$).

  Using this simple observation one may obtain the following property

  $$K_{D_1 \times D_2}((z_1, z_2), (w_1, w_2)) = K_{D_1}(z_1, w_1) K_{D_2}(z_2, w_2).$$
The importance of the Bergman kernel function

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- This kernel function can be used to construct a very useful automorphism-invariant metric, called the Bergman metric.

- Explicit formulas of the Bergman kernel function can help to solve important conjectures. For example, Mostow and Siu gave a counter-example to the important conjecture that the universal covering of a compact Kähler manifold of negative sectional curvature should be biholomorphic to the ball. In their counter-example, the explicit calculation of the Bergman kernel function and metric of the weak pseudo-convex domain \( \{ z = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^{14} + |z_2|^2 < 1 \} \) plays an essential role [see, for details, Mostow G D, Siu Y.-T., A compact Kähler surface of negative curvature not covered by the ball, Ann. of Math., 112(1980), 321-360].
The question is: For which domains can the Bergman kernel function be computed by explicit formulas?
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For Reinhardt domains it is a standard method for computing the Bergman kernel to use series representation, since we can choose $\phi_\alpha(z) = \frac{z^\alpha}{\|z^\alpha\|}$. 

Simple example: $D_{2,q,r} := \left\{ z \in \mathbb{C}^3 : |z_1|^2 + |z_2|^q < 1, |z_1|^2 + |z_3|^r < 1 \right\}$. It is well known, that function $f$ holomorphic in a Reinhardt domain $D \subset \mathbb{C}^n$ has a global expansion into a Laurent series $f(z) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha z^\alpha, z \in D$. Moreover if $D \cap (\mathbb{C}_j - 1 \times \{0\} \times \mathbb{C}^n - j) \neq \emptyset, j = 1,\ldots,n$, then $a_\alpha = 0$ for $\alpha \in \mathbb{Z}^n \setminus \mathbb{Z}^n_+$. Therefore $\{\phi_\alpha\}$ such that each $\alpha_i \geq 0$ is a complete orthogonal set for $L^2_a(\mathbb{C}^n)$. 

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$$D_{q,r}^2 := \{ z \in \mathbb{C}^3 : |z_1|^2 + |z_2|^q < 1, \quad |z_1|^2 + |z_3|^r < 1 \}$$
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Simple example:

\[
D_{q,r}^2 := \{ z \in \mathbb{C}^3 : |z_1|^2 + |z_2|^q < 1, \ |z_1|^2 + |z_3|^r < 1 \}
\]

Put \( \Phi_\alpha(z) = z_1^{\alpha_1}z_2^{\alpha_2}z_3^{\alpha_3} \). It is well known, that function \( f \) holomorphic in a Reinhardt domain \( D \subset \mathbb{C}^n \) has a global expansion into a Laurent series \( f(z) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha z^\alpha, \ z \in D \). Moreover if \( D \cap (\mathbb{C}^{j-1} \times \{0\} \times \mathbb{C}^{n-j}) \neq \emptyset, \ j = 1, \ldots, n \), then \( a_\alpha = 0 \) for \( \alpha \in \mathbb{Z}^n \setminus \mathbb{Z}_+^n \). Therefore \( \{ \Phi_\alpha \} \) such that each \( \alpha_i \geq 0 \) is a complete orthogonal set for \( L^2_a(D_{q,r}^2) \).
Proposition

Let \( \alpha_i \in \mathbb{Z}_+ \) for \( i = 1, 2, 3 \). Then, we have

\[
\| z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} \|^2_{L^2(D_{q,r}^2)} = \frac{\pi^3 \Gamma(\alpha_1 + 1) \Gamma\left(\frac{2\alpha_2+2}{q} + \frac{2\alpha_3+2}{r} + 1\right)}{(\alpha_2 + 1)(\alpha_3 + 1)\Gamma\left(\frac{2\alpha_2+2}{q} + \frac{2\alpha_3+2}{r} + \alpha_1 + 2\right)}
\]
Proposition

Let $\alpha_i \in \mathbb{Z}_+$ for $i = 1, 2, 3$. Then, we have

$$
\|z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}\|_{L^2(D_{q,r}^2)}^2 = \frac{\pi^3 \Gamma(\alpha_1 + 1) \Gamma(\frac{2\alpha_2+2}{q} + \frac{2\alpha_3+2}{r} + 1)}{(\alpha_2 + 1)(\alpha_3 + 1)\Gamma(\frac{2\alpha_2+2}{q} + \frac{2\alpha_3+2}{r} + \alpha_1 + 2)}
$$

Proof:

$$
\|z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}\|_{L^2(D_{q,r}^2)}^2 = \int_{D_{q,r}^2} |z_1|^{2\alpha_1} |z_2|^{2\alpha_2} |z_3|^{2\alpha_3} dV(z)
$$

we introduce polar coordinate in each variable by putting $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, $z_3 = r_3 e^{i\theta_3}$. After doing so, and integrating out the angular variables we have

$$(2\pi)^3 \int_0^1 \int_0^{(1-r_1^2)^{1/q}} \int_0^{(1-r_1^2)^{1/r}} r_1^{2\alpha_1+1} r_2^{2\alpha_2+1} r_3^{2\alpha_3+1} dr_1 dr_2 dr_3$$
\[ (2\pi)^3 \int_0^1 \int_0^1 \int_0^1 (1-r_1^2)^{1/q} (1-r_1^2)^{1/r} r_1^{2\alpha_1+1} r_2^{2\alpha_2+1} r_3^{2\alpha_3+1} \, dr_1 \, dr_2 \, dr_3 \]

Integrating out of \( r_2 \) and \( r_3 \) variables, we obtain

\[
\frac{(2\pi)^3}{(2\alpha_2 + 2)(2\alpha_3 + 2)} \int_0^1 r_1^{2\alpha_1+1} (1 - r_1^2)^{\frac{2\alpha_2+2}{q} + \frac{2\alpha_3+2}{r}} \, dr_1
\]

After little calculation using well known fact

\[
\int_0^1 x^a (1 - x^p)^b \, dx = \frac{\Gamma((a + 1)/p)\Gamma(b + 1)}{p\Gamma((a + 1)/p + b + 1)},
\]

we obtain desired result.
Now we discuss the Bergman kernel for $D_{q,r}^2$

**Theorem**

The Bergman kernel for

$$D_{q,r}^2 = \{ z \in \mathbb{C}^3 : |z_1|^2 + |z_2|^q < 1, \quad |z_1|^2 + |z_3|^r < 1 \}$$

is given by

$$K_{D_{q,r}^2}((z_1, z_2, z_3), (w_1, w_2, w_3)) =$$

$$qr(1 - \mu_2)(1 - \mu_3) + 2q(1 - \mu_2)(1 + \mu_3) + 2r(1 + \mu_2)(1 - \mu_3)$$

$$\pi^3 qr \frac{(1 - \nu_1)^{2+2/q+2/r}(1 - \mu_2)^3(1 - \mu_3)^3}{},$$

where $\nu_1 = z_1 \bar{w}_1$, $\nu_2 = z_2 \bar{w}_2$, $\nu_3 = z_3 \bar{w}_3$ and

$$\mu_2 = \frac{\nu_2}{(1 - \nu_1)^{2/q}}, \quad \mu_3 = \frac{\nu_3}{(1 - \nu_1)^{2/r}}.$$
Proof

By series representation of the Bergman kernel function, we have

\[
K_{D^2, q, r}((z_1, z_2, z_3), (w_1, w_2, w_3)) = \frac{1}{\pi^3} \sum_{\alpha_1, \alpha_2, \alpha_3=0}^{\infty} \frac{(\alpha_2 + 1)(\alpha_3 + 1)\Gamma\left(\frac{2\alpha_2 + 2}{q} + \frac{2\alpha_3 + 2}{r} + \alpha_1 + 2\right)}{\Gamma(\alpha_1 + 1)\Gamma\left(\frac{2\alpha_2 + 2}{q} + \frac{2\alpha_3 + 2}{r} + 1\right)} \nu_1^{\alpha_1} \nu_2^{\alpha_2} \nu_3^{\alpha_3},
\]

where \(\nu_1 = z_1 w_1\), \(\nu_2 = z_2 w_2\), \(\nu_3 = z_3 w_3\).

Sum out of \(\nu_1\) variable, we have

\[
\left(\frac{1}{1 - \nu_1}\right)^{2+2/q+2/r} \sum_{\alpha_2, \alpha_3=0}^{\infty} \frac{(\alpha_2 + 1)(\alpha_3 + 1)\Gamma\left(\frac{2\alpha_2 + 2}{q} + \frac{2\alpha_3 + 2}{r} + 2\right)}{\pi^3\Gamma\left(\frac{2\alpha_2 + 2}{q} + \frac{2\alpha_3 + 2}{r} + 1\right)} \mu_2^{\alpha_2} \mu_3^{\alpha_3},
\]

where \(\mu_2 = \frac{\nu_2}{(1-\nu_1)^{2/q}}\), \(\mu_3 = \frac{\nu_3}{(1-\nu_1)^{2/r}}\).
Proof continuation

\[
\left( \frac{1}{1 - \nu_1} \right)^{2+2/q+2/r} \sum_{\alpha_2,\alpha_3=0}^{\infty} \frac{(\alpha_2 + 1)(\alpha_3 + 1)\Gamma\left(\frac{2\alpha_2+2}{q} + \frac{2\alpha_3+2}{r} + 2\right)}{\pi^3\Gamma\left(\frac{2\alpha_2+2}{q} + \frac{2\alpha_3+2}{r} + 1\right)} \mu_2^{\alpha_2} \mu_3^{\alpha_3},
\]

Using the identity \( \Gamma(a + 1) = a\Gamma(a) \), after a little simplification, we obtain

\[
\sum_{\alpha_2,\alpha_3=0}^{\infty} \frac{(\alpha_2 + 1)(\alpha_3 + 1)\left(\frac{2\alpha_2+2}{q} + \frac{2\alpha_3+2}{r} + 1\right)}{\pi^3 (1 - \nu_1)^{2+2/q+2/r}} \mu_2^{\alpha_2} \mu_3^{\alpha_3}
\]

After a little calculations, we have

\[
\frac{qr(1 - \mu_2)(1 - \mu_3) + 2q(1 - \mu_2)(1 + \mu_3) + 2r(1 + \mu_2)(1 - \mu_3)}{\pi^3 qr (1 - \nu_1)^{2+2/q+2/r} (1 - \mu_2)^3 (1 - \mu_3)^3}.
\]
As a generalization of the Hermitian unit ball, we consider complex ellipsoids $\Omega_{p_1,\ldots,p_n}$ defined by

$$\Omega_{p_1,\ldots,p_n} = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^{2p_j} < 1 \right\},$$

where each $p_j$ is a positive real number.
A great interest in the theory of hypergeometric functions is motivated essentially by the fact that the solutions of many applied problems involving (for example) partial differential equations are obtainable with the help of such hypergeometric function (see, for details, [H.M. Srivastava, P.W. Karlsson, *Multiple Gaussian Hypergeometric Series*, 1985, p. 47]
A great interest in the theory of hypergeometric functions is motivated essentially by the fact that the solutions of many applied problems involving (for example) partial differential equations are obtainable with the help of such hypergeometric function (see, for details, [H.M. Srivastava, P.W. Karlsson, *Multiple Gaussian Hypergeometric Series*, 1985, p. 47]; see also the other works


For instance, the energy absorbed by some non-ferromagnetic conductor sphere included in an internal magnetic field can be calculated with the help of such functions.


Hypergeometric functions applications

Hypergeometric functions of several variables are used in physical and quantum chemical applications as well.


Multiple hypergeometric functions (that is, hypergeometric functions in several variables) occur naturally in a wide variety of problems.
In particular, the Lauricella function

\[ F_8(a, b_1, b_2, b_3; c_1, c_2; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}(b_1)_m(b_2)_n(b_3)_p}{(c_1)_m(c_2)_{n+p}m!n!p!} x^m y^n z^p, \]

and Appell’s functions \( F_A^{(n)} \) defined by

\[ F_A^{(n)}(a; b; c; x) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1} \cdots (b_n)_{m_n}}{m_1! \cdots m_n!(c_1)_{m_1} \cdots (c_n)_{m_n}} x_1^{m_1} \cdots x_n^{m_n}, \]

where \((a)_m = \Gamma(a + m)/\Gamma(a)\).
In 1996 Francsics and Hanges [G. Francsics, N. Hanges, *The Bergman kernel of complex ovals and multivariable hypergeometric functions*, J. Funct. Anal. 142 (1996) 494-510] expressed the Bergman kernel for complex ellipsoids in terms of Appell’s multivariable hypergeometric functions. In fact, the Bergman kernel $K(z, w)$ for $\Omega_{p_1, \ldots, p_n} := \{z \in \mathbb{C}^n : |z_1|^{2p_1} + \ldots + |z_n|^{2p_n} < 1\}$ is given by

$$
\prod_{j=1}^{n} p_j \frac{p_j - 1}{\pi^n} \sum_{k_1=0}^{p_1-1} \ldots \sum_{k_n=0}^{p_n-1} \frac{\Gamma \left( 1 + \sum_{j=1}^{n} \frac{k_j + 1}{p_j} \right)}{\prod_{j=1}^{n} \Gamma \left( \frac{k_j + 1}{p_j} \right)} (z \overline{w})^k \\
\cdot F_A^{(n)} \left( 1 + \sum_{j=1}^{n} \frac{k_j + 1}{p_j} ; 1 ; \frac{k + 1}{p} ; (z \overline{w})^p \right),
$$

where $(z \overline{w})^k = (z_1 \overline{w_1})^{k_1} \ldots (z_n \overline{w_n})^{k_n}$. 
The closed form of the Bergman kernel for the domain

$$\Omega_{m,n}^{(p,1)} := \{ (w, z) \in \mathbb{C}^m \times \mathbb{C}^n : \|w\|_m^{2p} + \|z\|_n^2 < 1 \}$$

which was firstly computed explicitly by D’Angelo in 1994. The formula is

**Theorem**

$$K_{\Omega_{m,n}^{(p,1)}} ((z, w); (\bar{z}, \bar{w})) = \sum_{k=0}^{n+1} \frac{d_k}{\pi^{m+n}p^{n}} \frac{\Gamma(m + k)(1 - \|z\|^2)^{-n-1+k/p}}{(1 - \|z\|^2)^{1/p} - \|w\|^2}^{m+k}$$

where the constants $d_k$ depend on $k, n, m, p$ and can be deduced by evaluating the parameter $y$ of the following equality at the negative integers

$$\prod_{l=0}^{n}(y + pl) = \sum_{k=-1}^{n} d_{k+1} \prod_{l=0}^{k}(y + l).$$
Explicit formulas

In 2008 J.-D. Park computed Bergman kernel for nonhomogeneous domain

\[ \Omega_{2,2} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^4 + |z_2|^4 < 1 \}. \]

**Theorem**

The Bergman kernel for \( \Omega_{2,2} \) is given by

\[
K_{\Omega_{2,2}}((z_1, z_2), (w_1, w_2)) = \frac{\nu_1(\pi + \arcsin \nu_1)f(\nu_1^2, \nu_2^2)}{\pi^3(1 - \nu_1^2)^{3/2}(1 - \nu_1^2 - \nu_2^2)^3} + \frac{\nu_2(\pi + \arcsin \nu_2)f(\nu_2^2, \nu_1^2)}{\pi^3(1 - \nu_2^2)^{3/2}(1 - \nu_1^2 - \nu_2^2)^3} + \frac{8\nu_1\nu_2}{\pi^2(1 - \nu_1^2 - \nu_2^2)^2} + \frac{2g(\nu_1^2, \nu_2^2)}{\pi^3(1 - \nu_1^2)(1 - \nu_2^2)(1 - \nu_1^2 - \nu_2^2)^2},
\]

where \( \nu_1 = z_1 \overline{w}_1, \nu_2 = z_2 \overline{w}_2, f(a, b) = 3(1 - a)^2 + 6b(1 - a) - b^2, \) and \( g(a, b) = 2 - a - b - (a - b)^2. \)
Explicit formulas

**Theorem**

For any positive real numbers $\lambda$ and $p$ the Bergman kernel for 
\[
\{ (z_1, z_2, w) \in \mathbb{C}^3 : \sqrt{|z_1|^{2p} + |z_2|^2 + |w|^2} < 1 \}
\]
is given by

\[
K((z_1, z_2, w), (\zeta_1, \zeta_2, \eta)) = \frac{((1 - \nu_3)^{\lambda} - \nu_2)^{\frac{1}{p} - 3}\nu_1^2(p - 1)(\lambda(p - 1) + p)}{(1 - \nu_3)^{2 - 2\lambda}\pi^3 p^2 (\nu_1 - ((1 - \nu_3)^{\lambda} - \nu_2)^{1/p})^4}
\]
\[
+ \frac{(1 - \nu_3)^{\lambda - 2}((1 - \nu_3)^{\lambda} - \nu_2)^{\frac{1}{p} - 3}\nu_1^2(p - 1)(\lambda - 1)\nu_2 p}{\pi^3 p^2 (\nu_1 - ((1 - \nu_3)^{\lambda} - \nu_2)^{1/p})^4}
\]
\[
+ \frac{((1 - \nu_3)^{\lambda} - \nu_2)^{\frac{3}{p} - 3}(p + 1)((1 - \nu_3)^{\lambda}(\lambda + \lambda p + p) + (\lambda - 1)\nu_2 p)}{(1 - \nu_3)^{2 - \lambda}\pi^3 p^2 (\nu_1 - ((1 - \nu_3)^{\lambda} - \nu_2)^{1/p})^4}
\]
\[
- \frac{((1 - \nu_3)^{\lambda} - \nu_2)^{\frac{2}{p} - 3}2\nu_1((1 - \nu_3)^{\lambda}(\lambda(p^2 - 2) + p^2) + (\lambda - 1)\nu_2 p^2)}{(1 - \nu_3)^{2 - \lambda}\pi^3 p^2 (\nu_1 - ((1 - \nu_3)^{\lambda} - \nu_2)^{1/p})^4},
\]

where $\nu_1 = z_1\bar{\zeta}_1$, $\nu_2 = z_2\bar{\zeta}_2$, $\nu_3 = w\bar{\eta}$. 
Explicit formulas

**Theorem**

For any positive real number \( \lambda \) the Bergman kernel for the domain \( D_1 = \{(z_1, z_2, w) \in \mathbb{C}^3 : \sqrt[3]{|z_1|^4 + |z_2|^4 + |w|^2} < 1\} \) is given by

\[
K((z_1, z_2, w), (\zeta_1, \zeta_2, \eta)) = \frac{24\lambda(1 - \nu_3)^{2\lambda-2}}{\pi^3 ((1 - \nu_3)^\lambda - \nu_1^2 - \nu_2^2)^3} \left( \frac{1}{\pi} + \frac{\nu_1\nu_2}{((1 - \nu_3)^\lambda - \nu_1^2 - \nu_2^2)} \right) + \frac{(8 - 8\lambda)(1 - \nu_3)^{\lambda-2}}{\pi^3 ((1 - \nu_3)^\lambda - \nu_1^2 - \nu_2^2)^2} \left( \frac{1}{\pi} + \frac{\nu_1\nu_2}{((1 - \nu_3)^\lambda - \nu_1^2 - \nu_2^2)} \right)
\]

\[
+ \frac{3\lambda}{\pi^4} \sum_{(i,j,k) \in I} \frac{c_{ijk}(1 - \nu_3)^{\lambda(i+j+k-1)-2}}{((1 - \nu_3)^\lambda - \nu_1^2 - \nu_2^2)^i ((1 - \nu_3)^\lambda - \nu_1^2)^j ((1 - \nu_3)^\lambda - \nu_2^2)^k}
\]

\[
+ \diamondsuit_{1,2} \left\{ \frac{3\lambda\nu_1 \sum_{i+j=3} c_{ij} \left( 1 - \frac{\nu_1^2}{(1-\nu_3)^\lambda} \right)^i \left( \frac{\nu_1^2}{(1-\nu_3)^\lambda} \right)^j \left( \pi + 2 \arcsin \frac{\nu_1}{\sqrt{(1-\nu_3)^\lambda}} \right) \right\}^{\frac{5}{2}}
\]

\[
+ \diamondsuit_{1,2} \left\{ \frac{(3 - 3\lambda)(1 - \nu_3)^{\lambda-2}\nu_1 \sqrt{(1 - \nu_3)^\lambda - \nu_1^2}}{\pi^3((1 - \nu_3)^\lambda - \nu_1^2 - \nu_2^2)^3} \left( 1 - \frac{\nu_1^4}{2 \left( (1-\nu_3)^\lambda - \nu_2^2 \right)^2} \right) \right\}^{\frac{5}{2}}
\]
Explicit formulas

Theorem

The Bergman kernel for \((z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^6 + |z_2|^2 < 1, \quad |z_1|^6 + |z_3|^2 < 1\) is given by

\[
K(z, w) = \frac{3}{2\pi^3} \frac{\partial^2}{\partial \nu_2 \partial \nu_3} \left\{ \nu_2 \nu_3 \sum_{j=0}^{3} \nu_1^j C_j F_8 \left( 3 + \frac{j}{3}, 1, 1; \frac{j}{3}, 3; \nu_1^3, \nu_2, \nu_3 \right) \right\},
\]

where \(C_i = \frac{\Gamma\left(3 + \frac{i}{3}\right)}{2\Gamma\left(\frac{i}{3}\right)}\), \(\nu_i = z_i \overline{w_i}\) for \(i = 1, 2, 3\).
Theorem

The Bergman kernel for \( \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^4 + |z_2|^4 < 1, \quad |z_1|^4 + |z_3|^4 < 1\} \) is given by

\[
K(z, w) = \frac{2}{\pi^3} \frac{\partial^2}{\partial \nu_2 \partial \nu_3} \left\{ \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} \sum_{k_3=0}^{1} c(k_1, k_2, k_3) \nu_1^{k_1} \nu_2^{k_2} \nu_3^{k_3} \right\} \cdot F_8 \left( 1 + \sum_{j=1}^{3} \frac{k_j + 1}{2}, 1, 1, 1; \frac{k_1 + 1}{2}, 1 + \frac{k_2 + 1}{2} + \frac{k_3 + 1}{2}; \nu_1^2, \nu_2^2, \nu_3^2 \right),
\]

where \( c(k_1, k_2, k_3) = \frac{\Gamma \left( 1 + \sum_{j=1}^{3} \frac{k_j + 1}{2} \right)}{\Gamma \left( \frac{k_1 + 1}{2} \right) \Gamma \left( 1 + \frac{k_2 + 1}{2} + \frac{k_3 + 1}{2} \right)}, \quad \nu_i = z_i \overline{w_i} \) for \( i = 1, 2, 3 \).
Explicit formulas

Theorem (K. Oeljeklaus, P. Pflug and E.H. Youssfi)

The Bergman kernel for minimal ball

\[ B_* := \{ z \in \mathbb{C}^n : \|z\|^2 + |z \cdot z| < 1 \}, \quad \text{where } z \cdot z := \sum_{j=1}^{n} z_j^2 \]

is given by the formula

\[ K(z, w) = \frac{\sum_{j=0}^{\left[ \frac{n}{2} \right]} \frac{(n+1)}{2j+1} X^{n-1-2j} Y^j (2nX - (n - 2j)(X^2 - Y))}{n(n+1)V(B_*)(X^2 - Y)^{n+1}}, \]

where \( X = 1 - \langle z, w \rangle \), \( Y = (z \cdot z)w \cdot w \), and \( V(B_*) \) is the Lebesgue volume of \( B_* \).
In 1966, Lu Qi-Keng [On Kähler manifolds with constant curvature, Chinese Math.-Acta, 8] asked: for which domains is the Bergman kernel function $K(z, w)$ zero-free? M. Skwarczyński called this problem Lu Qi-Keng conjecture in 1969 and gave the following definition:

**Definition**

A domain $D \subset \mathbb{C}^n$ is called a Lu Qi-Keng domain if $K_D(z, w) \neq 0$ for all $z, w \in D$. 
The Riemann mapping theorem characterizes the planar domains that are biholomorphically equivalent to the unit disk. In higher dimensions, there is no Riemann mapping theorem, and two natural problems arise.

1. Are there canonical representatives of biholomorphic equivalence classes of domains?
2. How can one tell that two particular domains are biholomorphically inequivalent?

As an approach to the first question, Stefan Bergman introduced the notion of a ”representative domain” to which a given domain may be mapped by ”representative coordinates”.
If $g_{jk}$ denotes the Bergman metric $\frac{\partial^2}{\partial z_j \partial \overline{z}_k} \log K(z, z)$, where $K$ is the Bergman kernel function, then the local representative coordinates based at the point $z_0$ are

$$\sum_{k=1}^{n} g_{kj}^{-1}(z_0) \frac{\partial}{\partial w_k} \log \frac{K(z, w)}{K(w, w)} \bigg|_{w=z_0}$$

These coordinates take $z_0$ to 0 and have complex Jacobian matrix at $z_0$ equal to the identity.

Zeroes of the Bergman kernel function $K(z, w)$ evidently pose an obstruction to the global definition of Bergman representative coordinates. This observation was Lu Qi-Keng’s motivation for asking which domains have zero-free Bergman kernel functions.
The Cartesian product of a Lu Qi-Keng domain with a Lu Qi-Keng domain is zero-free, and the Bergman kernel function of the Cartesian product domain of a non Lu Qi-Keng domain with any domain does have zeroes.

The range of the biholomorphic mapping of a Lu Qi-Keng domain is also a Lu Qi-Keng domain.

If $D_j$ form an increasing sequence whose union is $D$, and $K_j(z, w), K(z, w)$ are the Bergman kernel functions of $D_j, D$ respectively, it is well known that $\lim K_j(z, w) = K(z, w)$. Due to the Hurwitz’s theorem, if the Bergman kernel function of the limiting domain $D$ has zeroes, then so does the Bergman kernel function of the approximating $D_j$ when $j$ is sufficiently large.

The Bergman kernel function of the unit disk is evidently zero-free. Consequently, the Bergman kernel function of every simply connected planar domain $\neq \mathbb{C}$ is zero-free by the Riemann mapping theorem.
In 1969, M. Skwarczyński gave the first example that the Bergman kernel on an annulus in the complex plane $\Omega = \{ r < |z| < 1 \}$ has zeros if $0 < r < e^{-2}$.

In the same year Paul Rosenthal proved that the Bergman kernel function of annulus $D = \{0 < r < |z| < 1\}$ has zeroes; and for $k > 2$ there exists a bounded planar domain of connectivity $k$ which is not a Lu Qi-Keng domain.

1976, Nobuyuki Suita and Akira Yamada proved that the Bergman kernel function of every bounded, multiply connected, planar domain with smooth boundary has zeroes.

1986, Harold P. Boas proved that there exists a smooth bounded strongly pseudoconvex complete Reihardt domain in $\mathbb{C}^2$, whose Bergman kernel function has zeroes.
1998, P.Pflug and E.H.Youssfi proved that the minimal ball $B_* = \{ z \in \mathbb{C}^n : |z|^2 + |z \cdot z| < 1 \}$ is not Lu Qi-Keng domain if $n \geq 4$. Furthermore, when $0 < a \leq 1$, and $m$ is a sufficiently large integer, the interior approximating domain defined by

$$(|z|^2 + |z \cdot z|)^m + (|z|^2 - |z \cdot z|)^m + a^m|z|^{2m} < 1$$

is a concrete example of a smooth, strongly convex, algebraic domain whose Bergman kernel function has zeroes.

2000, Nguyễn Viêt Anh proved that there exists a strongly convex algebraic complete Reihardt domain which is not Lu Qi-Keng in $\mathbb{C}^n$ for any $n \geq 3$. 
1999, Harold P. Boas, Siqi Fu, Emil J. Straube proved that the Bergman kernel function of domain \( \{ z \in \mathbb{C}^2 : |z_1| + |z_2|^{2/p} < 1 \} \) has zeroes if \( p > 2 \). In the higher dimension they proved that the Bergman kernel function of convex domain \( \{ z \in \mathbb{C}^n : |z_1| + |z_2| + \ldots + |z_n| < 1 \} \) has zeros if and only if \( n \geq 3 \) (because if \( n = 2 \) its Bergman kernel function is zero-free). They proved also that the convex domain \( \{ z \in \mathbb{C}^n : |z_1| + |z_2|^2 + |z_3|^2 + \ldots + |z_n|^2 < 1 \} \) is not Lu Qi-Keng domain if and only if \( n \geq 4 \).
2006, B. Chen proved there exists a constant $n(p)$ depending on $p$ such that for all $n > n(p)$, the domain 
\[ \{|w|^{2p} + |z_1|^2 + \cdots + |z_n|^2 < 1\} \] is not Lu Qi-Keng.

2009, Liyou Zhang and Weiping Yin considered domains 
\[ \Omega^{(p,1)}_{m,n} := \{(w, z) \in \mathbb{C}^m \times \mathbb{C}^n : \|w\|^{2p}_m + \|z\|^2_n < 1\} \]

They proved that for $m$ and $n$ fixed, the zero set of the Bergman kernel for the domain $\Omega^{(p,1)}_{m,n}$ accumulates at a boundary point $(z_0, z_0) \in \partial \Omega^{(p,1)}_{m,n}$ when $p$ tends to infinity.
1996, Harold P. Boas proved that the bounded domains of holomorphy in $\mathbb{C}^n$ whose Bergman kernel functions are zero-free form a nowhere dense subset (with respect to a variant of the Hausdorff distance) of all bounded domains of holomorphy. Thus, contrary to former expectations, it is the normal situation for the Bergman kernel function of a domain to have zeroes.

The bounded homogeneous complete circular domains are always the Lu Qi-Keng domain.
Results in the case of Bergman kernel function is zero-free

- 2009, Liyou Zhang and Weiping Yin proved for fixed $n$ and $p$, there exists a constant $m_0 = m_0(n, p)$ such that the domain $\Omega_{m,n}^{(p,1)}$ is Lu Qi-Keng for all $m > m_0$.

- 2015, For any positive real numbers $q$ and $r$ the domain

$$D_{q,r}^2 = \{ z \in \mathbb{C}^3: |z_1|^2 + |z_2|^q < 1, \quad |z_1|^2 + |z_3|^r < 1 \}$$

is a Lu Qi-Keng domain.
Open problems

- Give necessary and sufficient conditions on an infinitely connected planar domain for its Bergman kernel function to have zeroes. For example, delete from the open unit disk a countable sequence of pairwise disjoint closed disks that accumulate only at the boundary of the unit disk. Does the Bergman kernel function of the resulting domain have zeroes?

- Characterize the vectors \((p_1, p_2, \ldots, p_n)\) of positive numbers for which the Bergman kernel function of the domain in \(\mathbb{C}^n\) defined by the inequality

\[
|z_1|^{2/p_1} + |z_2|^{2/p_2} + \cdots + |z_n|^{2/p_n} < 1
\]

is zero-free.

- It is an open question whether the three dimensional minimal ball is a Lu Qi-Keng domain.
Conclusion

It is a difficult problem to determine whether the Bergman kernel function of a specific domain has zeroes or not.

Unfortunately after almost 50 years there is no characterization of Lu Qi-Keng domains.
Intersection of two Lu Qi-Keng domains

Now we will consider following domains

\[ D_{q,r}^1 := \{ z \in \mathbb{C}^3 : |z_1| + |z_2|^q < 1, \quad |z_1| + |z_3|^r < 1 \}. \]

Similarly as in \( D_{q,r}^2 \) case, we have

**Proposition**

Let \( \alpha_i \in \mathbb{Z}_+ \) for \( i = 1, 2, 3 \). Then, we have

\[
\| z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} \|_{L^2(D_{q,r}^1)}^2 = \frac{2\pi^3 \Gamma(2\alpha_1 + 2) \Gamma(\frac{2\alpha_2 + 2}{q} + \frac{2\alpha_3 + 2}{r} + 1)}{(\alpha_2 + 1)(\alpha_3 + 1) \Gamma(\frac{2\alpha_2 + 2}{q} + \frac{2\alpha_3 + 2}{r} + 2\alpha_1 + 3)}. 
\]
Intersection of two Lu Qi-Keng domains

By series representation of the Bergman kernel function, we have

\[ K_{D_{q,r}^1}((0, z_2, z_3), (0, w_2, w_3)) = \]

\[ \frac{1}{2\pi^3} \sum_{\alpha_2, \alpha_3=0}^{\infty} \frac{(\alpha_2 + 1)(\alpha_3 + 1)\Gamma\left(\frac{2\alpha_2+2}{q} + \frac{2\alpha_3+2}{r} + 3\right)}{\Gamma(2)\Gamma\left(\frac{2\alpha_2+2}{q} + \frac{2\alpha_3+2}{r} + 1\right)} \nu_2^{\alpha_2} \nu_3^{\alpha_3}, \]

where \( \nu_2 = z_2 \overline{w}_2, \nu_3 = z_3 \overline{w}_3. \)

Using the identity \( \Gamma(a + 1) = a\Gamma(a) \), after a little calculation, we obtain

\[ K_{D_{q,r}^1}((0, z_2, z_3), (0, w_2, w_3)) = \frac{2r^2(\nu_2(\nu_2 + 4) + 1)(1 - \nu_3)^2}{\pi^3 q^2 r^2 (1 - \nu_2)^4 (1 - \nu_3)^4} \]

\[ + \frac{q^2(1 - \nu_2)^2 (-2r^2\nu_3 + (r - 2)(r - 1)\nu_3^2 + (r + 3)r + 8\nu_3 + 2)}{\pi^3 q^2 r^2 (1 - \nu_2)^4 (1 - \nu_3)^4} \]

\[ - \frac{qr (1 - \nu_2^2) (1 - \nu_3)(3r(\nu_3 - 1) - 4(\nu_3 + 1))}{\pi^3 q^2 r^2 (1 - \nu_2)^4 (1 - \nu_3)^4}. \]
Stability property for two-variable polynomial

Let us recall the stability property for two-variable polynomial with real coefficients. Consider a polynomial

\[ h(s, z) = \sum_{j=0}^{n} \sum_{k=0}^{m} h_{jk} s^j z^k \]

where \( s, z \in \mathbb{C} \) are complex variables, and for some \( j, k \) the coefficients \( h_{jn} \) and \( h_{mk} \) are not both zero.

**Definition**

A polynomial \( h(s, z) \) is said to satisfy the stability property if

\[ h(s, z) \neq 0, \quad (s, z) \in \overline{\mathbb{D}} \times \overline{\mathbb{D}}, \]

where \( \overline{\mathbb{D}} \) is the closure of \( \mathbb{D} \).
Intersection of two Lu Qi-Keng domains

**Proposition**

For any $r > 0$, domain $D^1_{r,r}$ defined by

$$D^1_{r,r} := \{ z = (z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1| + |z_2|^r < 1, \quad |z_1| + |z_3|^r < 1 \}$$

is not Lu Qi-Keng.

$$C_1 = \{ (z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1| + |z_2|^{2p} < 1, \quad z_3 \in \mathbb{D} \}$$

$$C_2 = \{ (z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1| + |z_3|^{2p} < 1, \quad z_2 \in \mathbb{D} \}$$
Intersection of two Lu Qi-Keng domains

**Proposition**

For any $r > 0$, domain $D_{r,r}^1$ defined by

$$D_{r,r}^1 := \{ z = (z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1| + |z_2|^r < 1, \quad |z_1| + |z_3|^r < 1 \}$$

is not Lu Qi-Keng.

**Lemma**

Intersection of two Lu Qi-Keng domains is not necessarily Lu Qi-Keng domain.

\[
C_1 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1| + |z_2|^{2p} < 1, \quad z_3 \in \mathbb{D}\}
\]

\[
C_2 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1| + |z_3|^{2p} < 1, \quad z_2 \in \mathbb{D}\}
\]
Thank you for your attention