Measures of Noncompactness, Some Applications and Visualisations

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1 Introduction

Measures of noncompactness are very useful tools in functional analysis, in particular, in the studies of

- metric fixed point theory
- the theory of operator equations in Banach spaces
- functional equations
- ordinary, partial and fractional differential and integral equations
- optimal control theory
- characterisations of compact operators between Banach spaces

The results presented here can be found in [10].
We give an axiomatic introduction to measures of noncompactness, and a survey of their most important basic properties.

We consider the Kuratowski, Hausdorff and separation measures of noncompactness.

Furthermore, we demonstrate how measures of noncompactness can be applied in fixed point theory, the theory of differential and integral equations, and the characterizations of classes of compact operators between certain Banach spaces.

Finally, we give some applications of our results and their visualisations in crystallography.
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2 Measures of Noncompactness of Sets

The first measure of noncompactness, denoted by $\alpha$, was defined and studied by Kuratowski [9] in 1930.

In 1955, G. Darbo [2] used the function $\alpha$ to prove his fixed point theorem which is a very important generalisation of Schauder’s fixed point theorem.

In 1957, Goldenštein, Go’hberg and Markus introduced and studied the Hausdorff or ball measure of noncompactness [4, 5].

We refer to [1, 7, 11, 15] for further studies.
We start with an axiomatic introduction to measures of noncompactness on bounded sets in complete metric spaces. Let $\mathcal{M}_X$ denote the class of all bounded subsets of a metric space $X$.

**Definition 2.1** Let $X$ be a complete metric space. A map $\phi : \mathcal{M}_X \rightarrow [0, \infty)$ is called a measure of noncompactness (MNC) on $X$ if it satisfies the following properties for all $Q, Q_1, Q_2 \in \mathcal{M}_X$

\begin{align*}
\text{(MNC.1)} & \quad \phi(Q) = 0 \text{ if and only if } Q \text{ is precompact} \quad \text{(Regularity)} \\
\text{(MNC.2)} & \quad \phi(Q) = \phi(\overline{Q}) \quad \text{(Invariance under closure)} \\
\text{(MNC.3)} & \quad \phi(Q_1 \cup Q_2) = \max\{\phi(Q_1), \phi(Q_2)\} \quad \text{(Semi–additivity)}.
\end{align*}
Basic Properties of an MNC

It is easy to see that the following basic results hold for MNC ([15, p. 19] for (2.4)).

**Proposition 2.2** Any MNC \( \phi \) satisfies following conditions for all \( Q, Q_1, Q_2 \in \mathcal{M}_X \)

\[
\begin{align*}
(2.1) & \quad Q_1 \subset Q_2 \implies \phi(Q_1) \leq \phi(Q_2) \quad \text{(Monotonicity)} \\
(2.2) & \quad \phi(Q_1 \cap Q_2) \leq \min\{\phi(Q_1), \phi(Q_2)\} \\
(2.3) & \quad \phi(Q) = 0 \quad \text{for every finite set } Q \quad \text{(Non–singularity)}.
\end{align*}
\]

If \( (Q_n) \) is a decreasing sequence of nonempty, closed sets in \( \mathcal{M}_X \) and \( \lim_{n \to \infty} \phi(Q_n) = 0 \), then

\[
(2.4) \quad \begin{cases} 
\bigcap_{n=1}^{\infty} Q_n \neq \emptyset \text{ is compact} \\
\text{(Cantor’s generalized intersection property).}
\end{cases}
\]
The discrete MNC

Throughout, let $(X, d)$ be a complete metric space.

**Example 2.3** The discrete MNC is the map $\phi_1 : \mathcal{M}_X \to [0, \infty)$ with

$$\phi_1(Q) = \begin{cases} 
0 & \text{if } Q \text{ is precompact} \\
1 & \text{otherwise}
\end{cases}$$

The Kuratowski MNC \[9\]

**Example 2.4** The Kuratowski MNC is the map $\alpha : \mathcal{M}_X \to [0, \infty)$ with

$$\alpha(Q) = \inf \left\{ \varepsilon > 0 : Q \subset \bigcup_{k=1}^n S_k, \ S_k \subset X, \ \text{diam}(S_k) < \varepsilon \ (k = 1, 2, \ldots, n \in \mathbb{N}) \right\}.$$
Example 2.5 *The Hausdorff or ball MNC (HMNC)* is the map

\[ \chi : \mathcal{M}_X \to [0, \infty) \]

such that

\[ \chi(Q) = \inf \left\{ \varepsilon > 0 : \exists k \in \mathbb{N} \quad Q \subset \bigcup_{k=1}^{n} B_{r_k}(x_k), \quad x_k \in X, \quad r_k < \varepsilon \right\}. \]
The separation MNC [6] or [15, Definition 3.1]

Example 2.6 A set $B$ in a metric space $(X, d)$ is said to be $r$–separated if $d(x, y) \geq r$ for all distinct $x, y \in B$, and the set $B$ is called an $r$–separation of $X$.

The separation MNC is the function $\beta : \mathcal{M}_X \rightarrow [0, \infty)$ with

$$\beta(Q) = \sup \{ r > 0 : Q \text{ has an infinite } r\text{–separation} \}$$

$$= \inf \{ r > 0 : Q \text{ does not have an infinite } r\text{–separation} \}$$

Remark 2.7 The functions $\alpha$, $\chi$ and $\beta$ are MNC’s in the sense of Definition 2.1 and so also satisfy (2.1)–(2.4) ([11, Lemmas 2.6, 2.11, Theorem 2.7] and [15, Remark 3.2]). Furthermore, they satisfy the following inequalities ([15, Remark 3.2])

$$\chi(Q) \leq \beta(Q) \leq \alpha(Q) \leq 2 \cdot \chi(Q) \text{ for all } Q \in \mathcal{M}_X.$$
Properties of $\alpha$, $\chi$ and $\beta$ in Banach spaces

Proposition 2.8 ([11, Theorems 2.8, 2.12]) Let $X$ be a Banach space, $Q, Q_1, Q_2 \in \mathcal{M}_X$ and $\psi$ be any of the functions $\alpha$ or $\chi$. Then we have
\begin{align*}
(2.5) & \quad \psi(Q_1 + Q_2) \leq \psi(Q_1) + \psi(Q_2), \\
(2.6) & \quad \psi(Q + x) = \psi(Q) \text{ for each } x \in X, \\
(2.7) & \quad \psi(\lambda Q) = |\lambda|\psi(Q) \text{ for each scalar } \lambda,
\end{align*}
and, if $\text{co}(Q)$ denotes the convex hull of $Q$,
\begin{equation}
(2.8) \quad \psi(Q) = \psi(\text{co}(Q)).
\end{equation}

If $X$ is infinite dimensional and $B_X$ and $S_X$ denote the open unit ball and the unit sphere in $X$, then ([15, Theorem 2.5, Corollary 2.6] or [11, Theorems 2.9, 2.14])
\begin{equation}
(2.9) \quad \alpha(B_X) = \alpha(S_X) = 2 \text{ and } \chi(B_X) = \chi(S_X) = 1.
\end{equation}

The separation $\text{MNC}$ satisfies (2.5) and (2.8) ([15, Theorems 3.4, 3.6]).
The Goldenštein, Go’hberg, Markus theorem \([4]\)

**Theorem 2.9** ([15, Theorem 4.2]) Let \(X\) be a Banach space with a Schauder basis. Then the function \(\mu : \mathcal{M}_X \rightarrow [0, \infty)\) with

\[
\mu(Q) = \limsup_{n \to \infty} \left( \sup_{x \in Q} \|R_n(x)\| \right)
\]

is an MNC on \(X\) which is invariant under the passage of the convex hull. Moreover, the following inequality holds for every \(Q \in \mathcal{M}_X\)

\[
\frac{1}{L} \cdot \mu(Q) \leq \chi(Q) \leq \inf_n \left( \sup_{x \in Q} \|R_n(x)\| \right) \leq \mu(Q),
\]

where \(L = \limsup_{n \to \infty} \|R_n\|\).
Another MNC $\nu$ can be defined by replacing $\limsup$ in the definition $\mu$ by $\liminf$.

**Theorem 2.10** ([15, Theorem 4.3]) Let $X$ be a Banach space with a Schauder basis. Then the function $\mu : M_X \to [0, \infty)$ with

$$
\nu(Q) = \liminf_{n \to \infty} \left( \sup_{x \in Q} \|R_n(x)\| \right)
$$

is an MNC on $X$ which is invariant under the passage of the convex hull. Moreover, the following inequality holds for every $Q \in M_X$

$$
\frac{1}{L} \cdot \nu(Q) \leq \chi(Q) \leq \nu(Q), \quad \text{where } L = \limsup_{n \to \infty} \|R_n\|.
$$
3 Measures of Noncompactness of Operators

In Section 2, we introduced some MNC’s on bounded sets of complete metric spaces and studied some of their properties.

Now we are going to define the MNC of operators between Banach spaces. This concept is can be applied in the characterization of compact linear operators.

We recall that an operator $L$ between Banach spaces $X$ and $Y$ is said to be compact, if its domain is all of $X$ and, for every bounded sequence $(x_n)$ in $X$, the sequence $(L(x_n))$ has a convergent subsequence in $Y$. 
MNC of Operators [11, Definition 2.24]

Let $\phi_1$ and $\phi_2$ be MNC’s on the Banach spaces $X$ and $Y$. An operator $L : X \rightarrow Y$ is said to be $(\phi_1, \phi_2)$–bounded if

$$L(Q) \in \mathcal{M}_Y \text{ for all } Q \in \mathcal{M}_X,$$

and there exists a non–negative real number $c$ such that

$$(3.1) \quad \phi_2(L(Q)) \leq c \phi_1(Q) \text{ for all } Q \in \mathcal{M}_X.$$

If an operator $L$ is $(\phi_1, \phi_2)$–bounded, then the number

$$\|L\|_{(\phi_1, \phi_2)} = \inf\{c \geq 0 : (3.1) \text{ holds}\}$$

is called the $(\phi_1, \phi_2)$–MNC of $L$.

If $\phi = \phi_1 = \phi_2$, then we write $\|L\|_{\phi} = \|L\|_{(\phi, \phi)}$, for short.
Theorem 3.1 ([11, Theorem 2.25, Corollary 2.26]) Let $X$, $Y$ and $Z$ be Banach spaces, $L \in \mathcal{B}(X, Y)$ and $\tilde{L} \in \mathcal{B}(Y, Z)$, $S_X$ and $\overline{B}_X$ be the unit sphere and the closed unit ball in $X$, and $\chi$ be the HMNC.

(a) Then we have

\begin{equation}
\|L\|_\chi = \chi(L(S_X)) = \chi(L(\overline{B}_X)).
\end{equation}

(b) Then $\| \cdot \|_\chi$ is a seminorm on $\mathcal{B}(X, Y)$, and

\begin{equation}
\|L\|_\chi = 0 \text{ if and only if } L \text{ is compact},
\end{equation}

$\|L\|_\chi \leq \|L\|$,

$\|\tilde{L} \circ L\|_\chi \leq \|\tilde{L}\|_\chi \cdot \|L\|_\chi$. 


4 Applications

Here we give some applications of MNC’s.

4.1 Darbo’s fixed point theorem

We use the Kuratowski MNC to prove Darbo’s fixed point theorem which is a generalisation of

Theorem 4.1 (Schauder’s fixed point theorem)

Let $C \neq \emptyset$ be a compact, convex subset of a Banach space $X$ and suppose that $T : C \to C$ is a continuous operator.

Then $T$ has a fixed point. ([15, Theorem 2.1])
**Darbo’s Fixed Point Theorem ([2] or [15, Theorem 5.4])**

**Theorem 4.2** Let $C \neq \emptyset$ be a bounded, closed and convex subset of a Banach space $X$, $\alpha$ be the Kuratowski MNC on $X$ and suppose that $T : C \rightarrow C$ is a continuous operator such that there exists a constant $c \in [0, 1)$ with $\alpha(T(Q)) \leq c \cdot \alpha(Q)$ for all $Q \subset S$. Then $T$ has a fixed point.

**Proof.** We put $C_0 = C$ and define a decreasing sequence of sets $C_{n+1} = \text{co}(T(C_n))$ for $n = 0, 1, \ldots$. By (2.8) and hypothesis

$$\alpha(C_{n+1}) = \alpha(\text{co}(T(C_n))) = \alpha(T(C_n)) \leq c \cdot \alpha(C_n) \leq c^{n+1} \cdot \alpha(C_0).$$

Since $c < 1$, we obtain $\lim_{n \to \infty} \alpha(C_n) = 0$. So $C_\infty = \bigcap_{n=1}^{\infty} C_n \neq \emptyset$ is compact by Cantor’s generalized intersection property (2.4). Since $C_\infty$ is also closed and convex, $T$ has a fixed point by Schauder’s fixed point theorem (Theorem 4.2). □
4.2 An application of Darbo’s fixed point theorem

An application of Darbo’s fixed point theorem to an initial value problem can be found in [14].

**Theorem 4.3 (Szulfa)** Let \( a, b \in \mathbb{R}^+ \), \( I = [t_0 - a, t_0 + a] \) and \( V = \overline{B}_r(x_0) \) be the closed ball of radius \( r \) and centre in \( x_0 \) in a Banach space \( X \), where \( t_0 \in \mathbb{R} \) and \( x_0 \in X \). If \( f : I \times V \to X \) is a continuous mapping such that for some constant \( c > 0 \) we have

\[
\alpha(f(I \times W)) \leq c \cdot \alpha(W) \quad \text{for any subset } W \text{ of } V,
\]

then there exists at least one solution of the initial value problem

\[
x'(t)f(t, x(t)), \quad x(t_0) = x_0
\]

defined on the interval \( J = [t_0 - h, t_0 + h] \), where

\[
0 < h \leq \min\{a, b/M, 1/c\} \quad \text{and } M = \sup\{\|f(t, x)\| : (t, x) \in I \times V\}.
\]
Outline of the Proof of Theorem 4.3

The integral equation

\begin{equation}
    x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) \, ds
\end{equation}

is equivalent to (4.1). We define the operator $T : C(J, X) \to C(J, X)$ by

$$
T_x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) \, ds \quad \text{for all } x \in C(J, X) \text{ and all } t \in J.
$$

Then $T$ is well defined and continuous and $C(J, V)$ is a closed, bounded and convex subset of $C(J, X)$. Also (4.2) becomes the operator equation $x = Tx$ for all $x \in C(J, V)$. It can be shown that $T(C(J, V))$ is a bounded and equicontinuous subset of $C(J, X)$ and $\alpha(T(Q)) < \alpha(Q)$ for all $Q \in M_{C(J, V)}$. The statement now follows from Darbo’s theorem.
4.3 Characterisation of some compact operators

Finally, we give an application of the HMNC to characterise bounded linear operators between certain sequence spaces.

Notations, BK and AK Spaces

We write \( \omega \) for the set of all complex sequences \( x = (x_k)_{k=1}^{\infty} \). A subspace \( X \) of \( \omega \) is said to be an \textbf{FK space} if it is a Fréchet space with continuous coordinates \( P_n : X \rightarrow \mathbb{C} \ (n \in \mathbb{N}) \), where \( P_n(x) = x_n \) for all \( x \in X \); a \textbf{BK} space is an \textbf{FK} space whose metric is given by a norm. An \textbf{FK} space \( X \) is said to have \textbf{AK} if \( x = \lim_{n \rightarrow \infty} \sum_{k=1}^{n} x_k e^{(k)} \), where, for each \( k \in \mathbb{N} \), \( e^{(k)} = (e^{(k)}_j)_{j=1}^{\infty} \) is the sequence with \( e^{(k)}_k = 1 \) and \( e^{(k)}_j = 0 \) for \( j \neq k \).

Let \( \ell_1 = \{ x \in \omega : \sum_{k=1}^{\infty} |x_k| < \infty \} \) be the set of all absolutely convergent series.
Theorem 4.4 ([11, Theorem 2.15]) We have

\[(4.3) \quad \chi(Q) = \lim_{n \to \infty} \left( \sup_{x \in Q} \sum_{k=n}^{\infty} |x_k| \right) \text{ for all } Q \in \mathcal{M}_{\ell_1}. \]

Proof. Since $\ell_1$ is a $BK$ space with $AK$ with $\|x\|_1 = \sum_{k=1}^{\infty} |x_k|$ by [16, Example 4.2.14], we can apply Theorem 2.9, and obtain by (2.10)

\[(4.4) \quad \mu(Q) = \lim_{n \to \infty} \left( \sup_{x \in Q} \sum_{k=n}^{\infty} |x_k| \right) \text{ for all } Q \in \mathcal{M}_{\ell_1}. \]

Clearly $\|R_n\| = 1$ for all $n \in \mathbb{N}$, so $L = \limsup_{n \to \infty} \|R_n\| = 1$. Now (4.3) follows from (4.4) and the inequalities in (2.10). \qed
The class \((X,Y)\)

If \(A = (a_{nk})_{n,k}^{\infty}\) is an infinite matrix of complex numbers and \(x \in \omega\), then we write \(A_n x = \sum_{k=1}^{\infty} a_{nk} x_k\) for \(n = 1, 2, \ldots\) and \(Ax = (A_n x)_n^{\infty}\) provided all the series converge. Let \(X\) and \(Y\) be subsets of \(\omega\) then we write \((X,Y)\) for the class of all infinite matrices \(A\) for which \(Ax \in Y\) for all \(x \in X\).

Relation between \((X,Y)\) and \(B(X,Y)\)

If \(X\) and \(Y\) are \(BK\) spaces, then \((X,Y) \subset B(X,Y)\), that is, every matrix \(A \in (X,Y)\) defines an operator \(L_A \in B(X,Y)\) by \(L_A(x) = Ax\) for all \(x \in X\) ([16, Theorem 4.2.8]).

If \(X\) has \(AK\), then \(B(X,Y) \subset (X,Y)\), that is, every operator \(L \in B(X,Y)\) is given by an infinite matrix \(A \in (X,Y)\) such that \(L(x) = Ax\) for all \(x \in X\) ([8, Theorem 1.9]).
Characterisation of compact operators in $\mathcal{B}(\ell_1)$

**Theorem 4.5 ([11, Corollary 2.29])** Let $L \in \mathcal{B}(\ell_1)$. Then $L$ is compact if and only if

$$(4.5) \quad \lim_{m \to \infty} \left( \sup_{k} \sum_{n=m}^{\infty} |a_{nk}| \right) = 0,$$

where $A$ is the infinite for which $L(x) = Ax$ for all $x \in \ell_1$.

**Proof.** Since $\ell_1$ is a $BK$ space with $AK$ with $\|x\|_1 = \sum_{k=1}^{\infty} |x_k|$ ($x \in \ell_1$), every $L \in \mathcal{B}(\ell_1)$ is given by a matrix $A \in (\ell_1, \ell_1)$. First we show

$$(4.6) \quad \|L\|_\chi = \lim_{m \to \infty} \left( \sup_k \sum_{n=m}^{\infty} |a_{nk}| \right).$$
Since $L \in B(\ell_1)$, we have

$$\|L(e^{(k)})\|_1 = \|Ae^{(k)}\|_1 = \sum_{n=1}^{\infty} |A_n e^{(k)}| = \sum_{n=1}^{\infty} |a_{nk}| \leq \|L\|$$

for all $k \in \mathbb{N}$, hence

$$\|A\| = \sup_k \sum_{n=1}^{\infty} |a_{nk}| \leq \|L\|. \hspace{1cm} (4.7)$$

Furthermore, we obtain for all $x \in \ell_1$

$$\|L(x)\|_1 = \sum_{n=1}^{\infty} |A_n x| \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |a_{nk}| \cdot |x_k|$$

$$= \sum_{k=1}^{\infty} |x_k| \left( \sum_{n=1}^{\infty} |a_{nk}| \right) \leq \|A\| \cdot \|x\|_1,$$

whence $\|L\| \leq \|A\|$. This and (4.7) yield $\|L\| = \|A\|$. 

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Now (4.3) and (3.2) imply

\[ \|L\|_\chi = \chi \left( L(S_{\ell_1}) \right) = \lim_{m \to \infty} \left( \sup_{x \in S_{\ell_1}} \|R_{m-1}(Lx)\|_1 \right). \]

Since \( R_m \in B(\ell_1) \), it follows as above that

\[ \sup_{x \in S_{\ell_1}} \|R_{m-1}(Lx)\|_1 = \sup_k \sum_{n=m}^{\infty} |a_{nk}|, \]

and we obtain the identity in (4.6).

Finally it follows from (4.6) and (3.3) that \( L \in B(\ell_1) \) is compact if and only if the condition in (4.5) holds. \( \square \)

**Remark 4.6** It is clear from the condition in (4.5) that the identity on \( \ell_1 \) is not compact.
5 Visualisations

We apply our software to visualise some neighbourhoods in certain real $FK$ and $BK$ spaces, and consider some applications to crystallography.

5.1 Visualisation of neighbourhoods

Let $n \in \mathbb{N}$ be given and $P_{k_1}, \ldots, P_{k_n} : \omega \to \mathbb{R}$ be the projections with $P_{k_j}(x) = x_{k_j}$ ($x = (x_k) \in \omega; j = 1, \ldots, n$). We obtain $\mathbb{R}^n = P(\omega) = (P_{k_1}(\omega), \ldots, P_{k_n}(\omega))$, and identify sequences with their projections into $n$–dimensional space.

We denote by $B_d(r, X_0) = \{X \in \mathbb{R}^n : d(X, X_0) < r\}$ the open ball in $(\mathbb{R}^n, d)$ of radius $r > 0$ with its centre in $X_0$, and consider the case $n = 3$ for the graphical representation of neighbourhoods by the boundaries $\partial B_d(X_0)$ of $B_d(r, X_0)$. 

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The space $\ell(p)$

**Example 5.1** Let $p = (p_k)_{k=1}^\infty$ be a real positive bounded sequence with $H = \sup_k p_k$, and $M = \max\{1, H\}$. We consider the set

$$\ell(p) = \left\{ x \in \omega : \sum_{k=0}^{\infty} |x_k|^{p_k} < \infty \right\},$$

if $p$ is a positive constant then this space reduces to the classical sequence spaces $\ell_p = \ell(p \cdot e)$. The set $\ell(p)$ is an $FK$ space with $AK$ with respect to its natural metric

$$d(p)(x, y) = \left( \sum_{k=1}^{\infty} |x_k - y_k|^{p_k} \right)^{1/M}.$$
Figure 5.1  $\partial B_{d(p)}(X_0, r)$ for

\[ p = (1/2, 2, 3/2); \quad p = (1/2, 4, 1/4) \]
5.2 Wulff’s crystals and potential surfaces

Here we deal with \textit{Wulff’s construction} and the graphical representation of \textit{Wulff’s crystals} and their surface energy functions as \textit{potential surfaces}. According to \textit{Wulff’s principle} \cite{Wulff1901}, the shape of a crystal is uniquely determined by its surface energy function, which is a real–valued function depending on a direction in space.

\textbf{Notations}

Let $\partial B^n$ denote the unit sphere in euclidean $\mathbb{R}^{n+1}$, and let $F : \partial B^n \rightarrow \mathbb{R}$ be a surface energy function. Then a natural representation of the function $F$ is given by

$$PM = \{ \vec{x} = F(\vec{e})\vec{e} \in \mathbb{R}^{n+1} : \vec{e} \in \partial B^n \}$$
Figure 5.2 *Potential surfaces and corresponding Wulff’s crystals*
Potential curve

If \( n = 1 \), then \( \vec{e} = \vec{e}(u) = (\cos u, \sin u) \) for \( u \in (0, 2\pi) \) and we obtain a potential curve with a parametric representation (left in Figure 5.3)

\[
PC = \{ \vec{x} = f(u)(\cos u, \sin u) : u \in (0, 2\pi) \}
\]

where \( f(u) = F(\vec{e}(u)) \).

Potential surface

If \( n = 2 \), then

\[
\vec{e} = \vec{e}(u^1, u^2) = (\cos u^1 \cos u^2, \cos u^1 \sin u^2, \sin u^1)
\]

for \( (u^1, u^2) \in R = (-\pi/2, \pi/2) \times (0, 2\pi) \)

and we obtain a potential surface with a parametric representation (right in Figure 5.3 and in Figure 5.4)

\[
PS = \{ \vec{x} = f(u^1, u^2)(\cos u^1 \cos u^2, \cos u^1 \sin u^2, \sin u^1) : (u^1, u^2) \in R \}
\]

where \( f(u^1, u^2) = F(\vec{e}(u^1, u^2)) \).
Figure 5.3 A potential curve and a potential surface

\[ f(u)\mathbf{e}(u) \]

\[ \mathbf{e}(u) = (\cos u, \sin u) \]

\[ F(\mathbf{e}(u^1, u^2))\mathbf{e}(u^1, u^2) \]
Figure 5.4 *Potential surfaces*
Wulff’s geometric principle of construction for crystals

**Theorem 5.2 (Wulff’s principle) ([17])** For every \( \vec{e} \in \partial B^n \), let \( E_{\vec{e}} \) denote the hyperplane orthogonal to \( \vec{e} \) and through the point \( P \) with position vector \( \vec{p} = F(\vec{e})\vec{e} \), and \( H_{\vec{e}} \) be the half space which contains the origin 0 and has the boundary \( E_{\vec{e}} = \partial H_{\vec{e}} \). Then the crystal \( C_F \) which has \( F \) as its surface energy function is uniquely determined and given by

\[
C_F = \bigcap_{\vec{e} \in \partial B^n} H_{\vec{e}} = \bigcap_{\vec{e} \in \partial B^n} \{ \vec{x} : \vec{x} \cdot \vec{e} \leq F(\vec{e}) \}
\]

(Left in in Figure 5.5).

**Remark 5.3** It is clear that if the surface energy function \( F \) is continuous then \( C_F \) is a closed convex subset of \( \mathbb{R}^n \).
Parametric representations for Wulff’s crystals

**Theorem 5.4** ([13, Theorem 5.3]) Let $F : \partial B^n \rightarrow \mathbb{R}^+$ be a continuous function. Then a point $X$ is on the boundary $\partial C_F$ of Wulff’s crystal $C_F$ corresponding to $F$ if and only if (right in Figure 5.5)

$$F(\vec{e}) \geq \vec{x} \cdot \vec{e} \text{ for all } \vec{e} \in \partial B^n \text{ and } F(\vec{e}_0) = \vec{x} \cdot \vec{e}_0 \text{ for some } \vec{e}_0 \in \partial B^n.$$ 

**Theorem 5.5** ([13, Theorem 5.4]) Let $F : \partial B^n \rightarrow \mathbb{R}^+$ be a continuous function and $C_F : \partial B^n \rightarrow \mathbb{R}^+$ be defined by

$$(5.1) \quad C_F(\vec{e}) = \inf \left\{ F(\vec{u})(\vec{e} \cdot \vec{u})^{-1} : \vec{u} \in \partial B^n \text{ and } \vec{e} \cdot \vec{u} > 0 \right\}.$$ 

Then the boundary $\partial C_F$ of Wulff’s crystal corresponding to $F$ is given by

$$(5.2) \quad \partial C_F = \{ \vec{x} = C_F(\vec{e})\vec{e} \in \mathbb{R}^{n+1} : \vec{e} \in \partial B^n \}.$$
**Case \( n = 2 \) of Theorem 5.5**

**Remark 5.6** We obtain for \( n = 2 \) (right in Figure 5.5)

\[
(5.3) \quad \bar{x}(u^1, u^2) = CF(\bar{e}'(u^1, u^2))\bar{e}'(u^1, u^2)
\]

for \((u^1, u^2) \in R = (-\pi/2, \pi/2) \times (0, 2\pi)\).

**Figure 5.5** Wulff’s constructions according to Theorems 5.2 and 5.4
Figure 5.6 Wulff’s constructions according to Theorems 5.4 and 5.5
The case of $F = \| \cdot \|$ 

**Corollary 5.7** ([13, Corollary 5.5]) Let $\| \cdot \|$ be a norm on $\mathbb{R}^{n+1}$ and, for each $\vec{w} \in \partial B^n$, let $\phi_{\vec{w}} : \mathbb{R}^{n+1} \to \mathbb{R}$ be defined by

$$
\phi_{\vec{w}}(\vec{x}) = \vec{w} \cdot \vec{x} = \sum_{k=1}^{n+1} w_k x_k \ (\vec{x} \in \mathbb{R}^{n+1}).
$$

Then the boundary $\partial C_{\| \cdot \|}$ of Wulff’s crystal corresponding to $\| \cdot \|$ is given by

$$
(5.4) \quad \partial C_{\| \cdot \|} = \left\{ \vec{x} = \frac{1}{\| \phi_{\vec{e}} \|} \cdot \vec{e} \in \mathbb{R}^{n+1} : \vec{e} \in \partial B^n \right\},
$$

where $\| \phi_{\vec{e}} \|^*$ is the norm of the functional $\phi_{\vec{e}}$, that is, the dual norm of $\| \cdot \|$. 

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A parametric representation for $\partial C\|\cdot\|$.

**Remark 5.8** If $n = 2$, we obtain from (5.4) and (5.3) the following parametric representation for Wulff’s crystal corresponding to a norm $\|\cdot\|$ in $\mathbb{R}^3$

\[
\hat{x}(u_1, u_2) = C\|\| \left( \hat{e}(u_1, u_2) \right) \cdot \hat{e}(u_1, u_2) = \frac{1}{\|\phi\hat{e}\|} \cdot \hat{e}(u_1, u_2)
\]

for $(u_1, u_2) \in \mathbb{R}$;

the potential surface has a parametric representation

\[
\hat{y} = F(\hat{e}(u_1, u_2)) \cdot \hat{e}(u_1, u_2) = \|\hat{e}(u_1, u_2)\| \cdot \hat{e}(u_1, u_2)
\]

for $(u_1, u_2) \in \mathbb{R}.$
Figure 5.7 Wulff’s crystals corresponding to the \( \ell_1 \) and \( \ell_\infty \) norms
The spaces and their norms in the next graphics were studied in [12, 13, 3].

\[ \text{The case } F = \| \cdot \|_{c_\infty(\Lambda)} \]

**Figure 5.8** Potential surface of the $c_\infty(\Lambda)$ norm and potential surface with corresponding Wulff’s crystal
The case $F = \| \cdot \|_{\mathcal{C}(\Lambda)}$

**Figure 5.9** Potential surface of the $\mathcal{C}(\Lambda)$ norm and potential surface with corresponding Wulff’s crystal
The case $F = \| \cdot \|_{v_{\infty}(\Lambda)}$

**Figure 5.10** Potential surface of the $v_{\infty}(\Lambda)$ norm and potential surface with corresponding Wulff’s crystal
The case $F = \| \cdot \|_{\mathcal{V}(\Lambda)}$

**Figure 5.11** Potential surface of the $\mathcal{V}(\Lambda)$ norm and potential surface with corresponding Wulff’s crystal
The case $F = \| \cdot \|_{w^p_\infty}$

**Figure 5.12** Potential surface of the $w^p_\infty$ norm and potential surface with corresponding Wulff’s crystal
The case $F = \| \cdot \|_{\mathcal{M}_p}$

Figure 5.13 Potential surface of the $\mathcal{M}_p$ norm and potential surface with corresponding Wulff’s crystal
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