İ s t a n b u l K ü l t ü r U n i v e r s i t y
Department of Computer Engineering

MAT 002 - Numerical Methods
Spring 2010-2011

First Midterm

March 17, 2011

Number: Name:

Directions – You have 120 minutes to complete the exam. Please do not leave the examination room in the first 30 minutes of the exam. There are six questions, of varying credit (100 points total). Indicate clearly your final answer to each question. You are allowed to use a calculator. During the exam, please turn off your cell phone(s). You cannot use the book or your notes. You have one page for "cheat-sheet" notes at the end of the exam papers. Do use the **radian mode** on your calculator when using the trigonometry buttons. Please use **five-decimal digit** in your calculations. The answer key to this exam will be posted on Department of Mathematics and Computer Science board after the exam.

Good luck! *Emel Yavuz Duman, PhD.*

Question 1.	
Question 2.	
Question 3.	

Question 4.	
Question 5.	
Question 6.	

Mark ________

Question 1. **Do one of (a), (b), (c)** 10 *points*

(a) Let $x = \frac{1}{3}$ and $y = \frac{4}{7}$. Find $x \oslash y = fl(fl(x) \div fl(y))$ by using 4-digit chopping arithmetic.
Answer. Since

$$x = \frac{1}{3} \Rightarrow fl(x) = 0.3333, \quad y = \frac{4}{7} \Rightarrow fl(y) = 0.5714$$

then we have

$$x \oslash y = fl(0.3333 \div 0.5714) = \underline{0.5833}.$$

(b) Use 3-digit rounding arithmetic to evaluate the polynomial $p(x) = x^3 - 4x^2 + 2x - 2.2$ at $x = 2.41$.
Answer. Since

x	x^2	x^3	$4x^2$	$2x$
2.41	5.81	14.0	23.2	4.82

then we have

$$p(2.41) = 14.0 - 23.2 + 4.82 - 2.2 = \underline{-6.58}.$$

(c) Find the base-10 decimal equivalent of the following number in IEEE 745 double precision format.

1	10000010001	1101000000000000000000000000000000000000000000000000

Answer. Since

$$s = 1$$
$$c = 2^0 + 2^4 + 2^{10} = 1041$$
$$f = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^4} = 0.5 + 0.25 + 0.0625 = 0.8125$$

then we have

$$(-1)^s 2^{c-1023}(1+f) = (-1)^1 2^{1041-1023}(1+0.8125) = -2^{18}(1.8125) = \underline{-475136}.$$

Question 2. 7 + 8 *points*

(a) Find an interval containing a solution of $\tan x - 2x + 1 = 0$.
Answer. Let $f(x) = \tan x - 2x + 1 = 0$. There are many intervals containing a solution of f. We show here only one of them. Since

$$f(1) = \tan 1 - 2(1) + 1 = 0.55740 > 0 \quad \text{and} \quad f(2) = \tan 2 - 2(2) + 1 = -5.1850 < 0$$

there exists a root of f between 1 and 2. So requested interval is $\underline{[1, 2]}$.

(b) How many bisection iterations would be required to locate this root in this interval to accuracy of $\varepsilon = 10^{-6}$.

Answer. Since $|p - p_n| \leq \frac{b-a}{2^n}$, then for $a = 1$, $b = 2$ we have

$$\frac{2-1}{2^n} \leq \varepsilon = 10^{-6} \Rightarrow 10^6 \leq 2^n \Rightarrow 6\log 10 \leq n\log 2 \Rightarrow \frac{6}{\log 2} \leq n \Rightarrow \frac{6}{0.30103} \leq n$$

$$\underline{n \geq 20}.$$

Question 3. 20 *points*

Use the Newton's Method to find a solution to within $\varepsilon = 10^{-4}$ for the function $f(x) = x^3 - e^{-x}$ where $0.5 \leq x \leq 1$, starting with $p_0 = 0.5$.

Answer. Since $f(x) = x^3 - e^{-x} \Rightarrow f'(x) = 3x^2 + e^{-x} \Rightarrow f'(0.5) = 1.3565 \neq 0$, thus we can use the Newton's Method. Since

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

we have following calculations:

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} = 0.5 - \frac{-0.48153}{1.3565} = 0.85498 \Rightarrow f(p_1) = f(0.85498) = 0.19969,$$

$$p_2 = p_1 - \frac{f(p_1)}{f'(p_1)} = 0.85498 - \frac{0.19969}{2.6183} = 0.77871 \Rightarrow f(p_2) = f(0.77871) = 0.013204$$

$$p_3 = p_2 - \frac{f(p_2)}{f'(p_2)} = 0.77871 - \frac{0.013204}{2.2782} = 0.77291 \Rightarrow f(p_3) = f(0.77291) = 6.0944 \times 10^{-5}$$

n	p_n	$f(p_n)$
0	0.5	-0.48153
1	0.85498	0.19969
2	0.77871	0.013204
3	0.77291	$6.0944 \times 10^{-5} \leq \varepsilon = 10^{-6}$

Therefore, requested root is <u>0.77291</u>.

Question 4. 15 *points*

Show that $g(x) = (3x + 19)^{\frac{1}{3}}$ has a unique fixed point in the interval $[0, \infty)$.

Hint: First show the existence and then the uniqueness.

Answer. Obviously $g(x) \in C[0, \infty)$ and $g([0, \infty)) = [19^{1/3}, \infty) \subset [0, \infty)$. So we have $g(x) \in [0, \infty)$. Therefore, there <u>exists</u> a fixed point of g in the interval $[0, \infty)$. Now, we want to show the <u>uniqueness</u> of that fixed point of g in $[0, \infty)$. We need to find an upper bound for $|g'(x)|$ which is less than 1:

$$|g'(x)| = \left|3\frac{1}{3}(3x+19)^{-2/3}\right| = \left|\frac{1}{(3x+19)^{2/3}}\right|$$
$$\leq \left|\frac{1}{(3(0)+19)^{2/3}}\right| = \frac{1}{19^{2/3}} = 0.14044 = k < 1.$$

Thus this fixed point is unique in $[0, \infty)$.

Question 5. *20 points*

Use four steps of the Bisection Method to find an approximate root of $\sin x = 0.8x$ starting with $a_0 = 1$ and $b_0 = 1.5$.

Answer. Let $f(x) = \sin x - 0.8x = 0$. Since $f(a_0) = f(1) = \sin(1) - 0.8(1) = 0.041471 > 0$ and $f(b_0) = f(1.5) = \sin(1.5) - 0.8(1.5) = -0.20251 < 0$, this means $f(1)f(1.5) < 0$, there exist a roof of f in $[1, 1.5]$. So we can use the Bisection Method:

n	a_n	b_n	p_n	$f(p_n)$
0	1	1.5	1.25	−0.051015
1	1	1.25	1.125	0.0022676
2	1.125	1.25	1.1875	−0.022563
3	1.125	1.1875	1.1563	−0.0097207

So the root is approximately $\underline{1.1563}$.

Question 6. *10 + 10 points*

(a) Find the second Taylor Polynomial $P_2(x)$ for the function $f(x) = xe^x + x$ about $x_0 = 0$, and use $P_2(0.2)$ to approximate $f(0.2)$.

Answer. Since $P_2(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2$, we have

$$f(x) = xe^x + x \Rightarrow f(0) = 0$$
$$f'(x) = e^x + xe^x + 1 \Rightarrow f'(0) = 2$$
$$f''(x) = e^x + e^x + xe^x \Rightarrow f''(0) = 2$$

$$P_2(x) = 0 + \frac{2}{1!}x + \frac{2}{2!}x^2 \Rightarrow \underline{P_2(x) = 2x + x^2}.$$

Also, we know that $f(0.2) = P_2(0.2) + R_2(0.2)$ where R_2 denotes the remainder of f. Thus, we deduce that $f(0.2) \approx P_2(0.2)$ which is

$$f(0.2) \approx P_2(0.2) = 2(0.2) + (0.2)^2 = \underline{0.44}.$$

(b) Find the absolute error and relative error in (a).

Answer. $f(0.2) = (0.2)e^{(0.2)} + (0.2) = 0.44428$

Absolute Error : $|f(0.2) - P_2(0.2)| = |R_2(0.2)| = |0.44428 - 0.44| = \underline{0.00428}$,

Relative Error : $\dfrac{|f(0.2) - P_2(0.2)|}{|f(0.2)|} = \dfrac{|R_2(0.2)|}{|f(0.2)|} = \dfrac{|0.44428 - 0.44|}{0.44428} = \underline{0.96336 \times 10^{-2}}$.