İ s t a n b u l K ü l t ü r U n i v e r s i t y
Department of Computer Engineering

MAT 002 - Numerical Methods
Spring 2011-2012

Final Exam

May 22, 2012

SOLUTIONS

Directions – You have 110 minutes to complete the exam. Please do not leave the examination room in the first 30 minutes of the exam. There are five questions, of varying credit (100 points total). Indicate clearly your final answer to each question. You are allowed to use a calculator. During the exam, please turn off your cell phone(s). You cannot use the book or your notes. You have one page for "cheat-sheet" notes at the end of the exam papers. Do use the **radian mode** on your calculator when using the trigonometry buttons. Please use **five-decimal digit** in your calculations.

Good luck! *Emel Yavuz Duman, PhD.*

Question 1.		Question 4.	
Question 2.		Question 5.	
Question 3.		MARK	

Question 1. 15 *points*

From the following table, find the number of students who obtain less than 90 mark by using *Newton's Backward Difference Formula.*

Marks	$20-40$	$40-60$	$60-80$	$80-100$
Number of students	11	23	13	8

Answer.

Marks less than x	No. of Students f	∇f	$\nabla^2 f$	$\nabla^3 f$
$x_3 = 100$	$55 = f(x_3)$			
$x_2 = 80$	47	$8 = \nabla f(x_3)$		
$x_1 = 60$	34	13	$-5 = \nabla^2 f(x_3)$	
$x_0 = 40$	11	23	-10	$5 = \nabla^3 f(x_3)$

Since $s = \dfrac{x - x_n}{h} = \dfrac{90-100}{20} = -0.5$, then

$$\begin{aligned} P_3(90) &= f(100) + s \nabla f(100) + \frac{s(s+1)}{2!} \nabla^2 f(100) + \frac{s(s+1)(s+2)}{3!} \nabla^3 f(100) \\ &= 55 + (-0.5)8 + \frac{(-0.5)(-0.5+1)}{2!}(-5) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}5 \\ &= 51.313. \end{aligned}$$

Hence the number of students get the mark less than 90 is 52.

Question 2. $5+15$ *points*

Consider the equation $e^x - x^2 + 3x - 2 = 0$.

(a) Find an interval $[a, b]$ which contains a root of this equation.

Answer. Let we take $[a, b] = [0, 1]$. Since

$$f(0) = -1 < 0 \quad \text{and} \quad f(1) = 2.7183 > 0,$$

then there exists a root of $f(x) = e^x - x^2 + 3x - 2$ in the interval $[0, 1]$.

(b) Calculate the fourth root approximation p_4, by using any root finding algorithm in the interval $[a, b]$ determined in (a).

Answer. We use the Bisection algorithm

a_n	b_n	p_n	$f(p_n)$
0	1	0.5	0.89872
0	0.5	0.25	-0.028475
0.25	0.5	0.375	0.43937
0.25	0.375	0.3125	0.20668

Therefore, $p_4 \approx 0.3125$.

Question 3. 15 *points*

The function $f(x) = \sin x$ will be approximated at $x = \frac{\pi}{2}$ with the help of the third order *Lagrange interpolating polynomial* by using the nodes

$$x_0 = \frac{\pi}{2} - 2h, \quad x_1 = \frac{\pi}{2} - h, \quad x_2 = \frac{\pi}{2} + h, \quad x_3 = \frac{\pi}{2} + 2h.$$

If the involved error in this approximation will be less than 10^{-2}, calculate an upper bound for h.

Answer. Since the error in the Lagrange interpolating polynomial is given by

$$E = \left| \frac{f^{(n+1)}(z)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n) \right|$$

where z is a number between x_0 and x_n, then for $n = 3$ we have

$$f(x) = \sin x,\ f'(x) = \cos x,\ f''(x) = -\sin x,\ f'''(x) = -\cos x,\ f^{(iv)}(x) = \sin x \Rightarrow$$

$$E = \left| \frac{\sin z}{4!} \left(\frac{\pi}{2} - \frac{\pi}{2} + 2h\right) \left(\frac{\pi}{2} - \frac{\pi}{2} + h\right) \left(\frac{\pi}{2} - \frac{\pi}{2} - h\right) \left(\frac{\pi}{2} - \frac{\pi}{2} - 2h\right) \right| = \frac{4h^4}{4!} \underbrace{|\sin z|}_{\leqq 1}$$

$$\leqq \frac{h^4}{6} \leqq 10^{-2} \Rightarrow h^4 \leqq 6 \cdot 10^{-2} \Rightarrow h \leqq 0.49492.$$

Question 4. $10 + 10 + 5$ *points*

Consider the definite integral $\int_1^3 \{(x+2)^4 + e^x\}dx$. It is desired to approximate it within 10^{-1} accuracy using the Composite Simpson's rule.

(a) Determine the possible smallest n value. Where n is the number of the subintervals.

Answer. Since the error in the Composite Simpson' Rule is given by

$$E = \left| \frac{b-a}{180} h^4 f^{(iv)}(z) \right|$$

where z is a number between a and b, then for $b = 3$, $a = 1$, $h = \frac{b-a}{n} = \frac{2}{n}$, $f(x) = (x+2)^4 + e^x$, $f'(x) = 4(x+2)^3 + e^x$, $f''(x) = 12(x+2)^2 + e^x$, $f'''(x) = 24(x+2) + e^x$ and $f^{(iv)}(x) = 24 + e^x$ we have

$$E = \left| \frac{3-1}{180} \left(\frac{2}{n}\right)^4 (24 + e^z) \right| \leq \frac{2^5}{180} \frac{1}{n^4} \max_{1 \leqq z \leqq 3} |24 + e^z| = \frac{2^5}{180} \frac{1}{n^4} |24 + e^3|$$

$$\leqq 7.8374 \frac{1}{n^4} \leqq 10^{-1} \Rightarrow n^4 \geqq 78.374 \Rightarrow n \geqq 2.9754.$$

Since n should be an even number, we see that $n \geqq 4$.

(b) Calculate the approximation by using the n value that you have found in (a).

Answer. $n = 4$, $h = \frac{3-1}{4} = 0.5$, $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2.0$, $x_3 = 2.5$, $x_4 = 3.0$

$$\begin{aligned} \int_1^3 \{(x+2)^4 + e^x\}dx &\approx \frac{0.5}{3}[f(x_0) + 2f(x_2) + 4\{f(x_1) + f(x_3)\} + f(x_4)] \\ &\approx \frac{0.5}{3}[f(1) + 2f(2) + 4\{f(1.5) + f(2.5)\} + f(3)] \\ &\approx 593.79. \end{aligned}$$

(c) Evaluate the actual error, is it less than 10^{-1}.

Answer.

$$\int_1^3 \{(x+2)^4 + e^x\}dx = \left[\frac{(x+2)^5}{5} + e^x\right]_1^3 = \frac{5^5}{5} + e^3 - \frac{3^5}{5} - e = 593.77$$

$$\text{Absolute Error:} |593.77 - 593.79| = 2 \times 10^{-2} < 10^{-1}.$$

Question 5. 10 + (10 + 5) *points*

(a) Use the most accurate three-point formula to determine $f'(3.1)$ in the following table.

x	3.0	3.1	3.2	3.3
$f(x)$	417.43	507.36	617.09	750.99

Answer. We use midpoint formula with $h = 0.1$:

$$f'(3.1) \approx \frac{1}{2h}[f(x_0+h)-f(x_0-h)] = \frac{1}{2(0.1)}[f(3.2)-f(3.0)] = \frac{1}{0.2}[617.09-417.43] = 998.3$$

(b) The data in (a) were taken from the function $f(x) = e^{2x} + x^2 + 5$. Compute the actual error in the approximation and also find an error bound for this approximation.
Answer. $f'(x) = 2e^{2x} + 2x \Rightarrow f'(3.1) = 2e^{2(3.1)} + 2(3.1) = 991.70.$

$$\text{Actual Error:}|991.70 - 998.3| = 6.6.$$

$f''(x) = 4e^{2x} + 2,\ f'''(x) = 8e^{2x}$

$$E = \left|\frac{h^2}{6}f'''(z)\right| = \left|\frac{0.1^2}{6}8e^{2z}\right| \leqq \frac{0.1^2 8}{6}\max_{3\leqq z\leqq 3.2}|e^{2z}| = \frac{0.1^2 8}{6}e^{2(3.2)} = 8.0246 < 6.6.$$