İ s t a n b u l K ü l t ü r U n i v e r s i t y
Department of Computer Engineering

MAT 002 - Numerical Methods
Fall 2011-2012

First Midterm

SOLUTIONS

March 22, 2012

Directions – You have 110 minutes to complete the exam. Please do not leave the examination room in the first 30 minutes of the exam. There are five questions, of varying credit (100 points total). Indicate clearly your final answer to each question. You are allowed to use a calculator. During the exam, please turn off your cell phone(s). You cannot use the book or your notes. You have one page for "cheat-sheet" notes at the end of the exam papers. Do use the **radian mode** on your calculator when using the trigonometry buttons. Please use **five-decimal digit** in your calculations. The answer key to this exam will be posted on Department of Mathematics and Computer Science board after the exam.

Good luck! *Emel Yavuz Duman, PhD.*

Question 1.			Question 4.	
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Question 1. $5 + 5 + 10$ *points*

(a) Show that the equation $x = 3 + \frac{\sin x}{2}$ has a solution in the interval $[3, 4]$.

Answer. Let $f(x) = x - 3 - \frac{\sin x}{2}$. It clear that f is a continuous function on $[3, 4]$, and

$$f(3) = 3 - 3 - \frac{\sin(3)}{2} = -0.070560 < 0, \text{ and } f(4) = 4 - 3 - \frac{\sin(4)}{2} = 1.3784 > 0$$

i.e, $f(3)$ and $f(4)$ have opposite sings. So, the Intermediate Value Theorem implies that a number p exists in $(3, 4)$ with $f(p) = 0$.

(b) Determine the number of iterations necessary to solve the equation $x = 3 + \frac{\sin x}{2}$ with accuracy 10^{-1} using $a_1 = 3$ and $b_1 = 4$.

Answer. We know that the Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with $|p_n - p| \leqq \frac{b-a}{2^n}$, when $n \geqq 1$. Therefore, we use logarithms to find an integer n, that satisfies $\frac{b-a}{2^n} \leqq 10^{-1}$:

$$\frac{4-3}{2^n} \leqq 10^{-1} \Rightarrow 2^{-n} \leqq 10^{-1} \Rightarrow 2^n \geqq 10^1 \Rightarrow n \log(2) \geqq 1 \Rightarrow n \geqq \frac{1}{\log(2)} = 3.3219.$$

Hence, 4 iterations will ensure an approximation accurate to within 10^{-1}.

(c) Use the Bisection method to determine an approximation to the solution that is accurate to at least within 10^{-1} for the equation $x = 3 + \frac{\sin x}{2}$ where $3 \leqq x \leqq 4$.

Answer.

n	a_n	b_n	p_n	$f(p_n)$
1	3	4	3.5	0.67539
2	3	3.5	3.25	0.30410
3	3	3.25	3.125	0.11670
4	3	3.125	3.0625	$0.22995 \times 10^{-1} < 10^{-1}$

So, we see that $p \approx p_4 = 3.0625$.

Question 2. 5 + 15 *points*

(a) Find an interval $[a, b]$ containing a solution of $f(x) = \sin x - \frac{1}{10}\ln x$.

Answer. Since the domain of the logarithmic function is the set of strictly positive real numbers, we only focus on the interval $(0, \infty)$. Although, there are many intervals containing a solution of f, since

$$f(3) = \sin(3) - \frac{1}{10}\ln(3) = 0.031259 > 0 \text{ and } f(4) = \sin(4) - \frac{1}{10}\ln(4) = -0.89543 < 0,$$

then, by the Intermediate Value Theorem there exists a solution of continuous function f in $(3, 4)$.

(b) Use four steps of the Secant method to find a solution of $f(x) = \sin x - \frac{1}{10}\ln x$ in this interval determined in (a) by taking $p_0 = a$ and $p_1 = b$.

Answer. Let $p_0 = 3$ and $p_1 = 4$. By applying the Secant method formula

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$$

we obtain the following table:

n	p_n	$f(p_n)$
0	3	0.031259
1	4	-0.89543
2	3.0337	-3.2949×10^{-3}
3	3.0301	4.0225×10^{-4}

$$f(p_0) = f(3) = 0.031259, \ f(p_1) = f(4) = -0.89543,$$

$$p_2 = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)} = 4 - \frac{(-0.89543)(4 - 3)}{-0.89543 - 0.031259} = 3.0337,$$

$$f(p_2) = f(3.0337) = \sin(3.0337) - \frac{1}{10}\ln(3.0337) = -3.2949 \times 10^{-3},$$

$$p_3 = p_2 - \frac{f(p_2)(p_2 - p_1)}{f(p_2) - f(p_1)} = 3.0337 - \frac{(-3.2949 \times 10^{-3})(3.0337 - 4)}{-3.2949 \times 10^{-3} + 0.89543} = 3.0301,$$

$$f(p_3) = f(3.0301) = \sin(3.0301) - \frac{1}{10}\ln(3.0301) = 4.0225 \times 10^{-4}.$$

So, we see that $p \approx p_3 = 3.0301$.

Question 3. 15 *points*

Use the Newton's method to approximate the root of function f given by

$$f(x) = \tan x + \ln x$$

by taking $p_0 = 0.5$, within $\varepsilon = 10^{-4}$.
Hint. $\frac{d}{dx}\tan x = \frac{1}{\cos^2 x}$.
Answer. Since

$$f'(x) = \frac{1}{\cos^2 x} + \frac{1}{x} \Rightarrow f'(0.5) = \frac{1}{\cos^2(0.5)} + \frac{1}{0.5} = 3.2984 \neq 0,$$

we can apply the Newton's method. By using the Newton's method formula

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

we obtain the following table:

n	p_n	$f'(p_n)$	$f(p_n)$
0	0.5	3.2984	-0.14684
1	0.54452	3.2032	-0.22601×10^{-2}
2	0.54523		$0.13713 \times 10^{-4} < 10^{-4}$

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} = 0.5 - \frac{-0.14684}{3.2984} = 0.54452,$$

$$f'(0.54452) = \frac{1}{\cos^2(0.54452)} + \frac{1}{0.54452} = 3.2032 \neq 0,$$

$$f(0.54452) = \tan(0.54452) + \ln(0.54452) = -0.22601 \times 10^{-2},$$

$$p_2 = p_1 - \frac{f(p_1)}{f'(p_1)} = 0.54452 - \frac{-0.22601 \times 10^{-2}}{3.2032} = 0.54523,$$

$$f(0.54523) = \tan(0.54523) + \ln(0.54523) = 0.13713 \times 10^{-4}.$$

So, we see that $p \approx p_2 = 0.54523$.

Question 4. $15 + (2 + 8)$ *points*

Let $f(x) = e^x \cos x$ and $x_0 = 0$.

(a) Find the third Taylor polynomial $P_3(x)$ and remainder $R_3(x)$ for f about $x_0 = 0$.

Answer. Since

$$f(x) = e^x \cos x \Rightarrow f(0) = e^0 \cos 0 = 1,$$

$$f'(x) = e^x \cos x - e^x \sin x = e^x(\cos x - \sin x) \Rightarrow f'(0) = e^0(\cos 0 - \sin 0) = 1,$$

$$f''(x) = e^x(\cos x - \sin x) + e^x(-\sin x - \cos x) = -2e^x \sin x \Rightarrow f''(0) = -2e^0 \sin 0 = 0,$$

$$f'''(x) = -2(e^x \sin x + e^x \cos x) = -2e^x(\sin x + \cos x) \Rightarrow f'''(0) = -2e^0(\sin 0 + \cos 0) = -2,$$

$$f^{(iv)}(x) = -2[e^x(\sin x + \cos x) + e^x(\cos x - \sin x)] = -4e^x \cos x \Rightarrow f^{(iv)}(z(x)) = -4e^{z(x)} \cos(z(x))$$

where $z(x)$ is a number between $x_0 = 0$ and x, then

$$\begin{aligned} f(x) = e^x \cos x &= P_3(x) + R_3(x) \\ &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(iv)}(z(x))}{4!}(x-0)^4 \\ &= 1 + x + \frac{-2}{3!}x^3 + \frac{-4\cos(z(x))e^{z(x)}}{4!}x^4 \\ &= 1 + x - \frac{x^3}{3} - \frac{\cos(z(x))e^{z(x)}}{6}x^4. \end{aligned}$$

(b) Use $P_3(0.05)$ to approximate the value of $f(0.05)$ and find an upper bound for this approximation by using $R_3(0.05)$.

Answer. Since $P_3(x) = 1 + x - \frac{x^3}{3}$, we have

$$f(0.05) \approx P_3(0.05) = 1 + 0.05 - \frac{0.05^3}{3} = 1.0500.$$

Next, since $R_3(x) = -\frac{\cos(z(x))e^{z(x)}}{6}x^4$, we obtain that

$$R_3(0.05) = -\frac{\cos(z(0.05))e^{z(0.05)}}{6}0.05^4$$

where $0 \leqq z(0.05) \leqq 0.05$. By using the fact that $|\cos x| \leqq 1$ for all real x, we see that

$$\begin{aligned} |R_3(0.05)| &= \left| -\frac{\cos(z(0.05))e^{z(0.05)}}{6}0.05^4 \right| = 1.0417 \times 10^{-5}|e^{z(0.05)}||\cos(z(0.05))| \\ &\leqq 1.0417 \times e^{0.05} \times 10^{-5} \times e^{0.05} = 1.0951 \times \times 10^{-6}. \end{aligned}$$

Question 5. $10 + 10$ *points*

(a) Show that $g(x) = \ln(2x+1)$ has a fixed point in the interval $[1, 2]$. (Do not approximate the fixed point. Verify all conditions)

Answer. Since $x \in [1, 2]$, then $2x + 1 > 0$. Thus, logarithmic function $g(x) \in C[1, 2]$ for all $x \in [1, 2]$. Also, since

$$g'(x) = \frac{2}{2x+1} > 0 \text{ for all } x \in [1, 2],$$

we see that g is monotonic increasing. Thus, it attains its maximum and minimum values at the end points. As an increasing function, g has a minimum at the left hand side of the interval and it satisfies

$$g(1) = \ln(3) = 1.0986 > 1.$$

Furthermore, g has a maximum at the right hand side of the interval and it satisfies

$$g(2) = \ln(5) = 1.6094 < 2.$$

Therefore, we see that $g(x) \in [1, 2]$ for all $x \in [1, 2]$. Consequently, by the Fixed Point Theorem, there exists at least one fixed point of g in $[1, 2]$. Next, we show the uniqueness of this fixed point in the given interval. For this purpose, we need to find an upper bound k for $|g'(x)|$ which is less than 1:

$$|g'(x)| = \left|\frac{2}{2x+1}\right| \leqq \max_{1 \leqq x \leqq 2} \left|\frac{2}{2x+1}\right| = \frac{2}{2 \cdot 1 + 1} = \frac{2}{3} = k < 1.$$

So, this fixed point in [1,2] of g is unique.

(b) How many fixed point iteration would be required to locate this fixed point in $[1, 2]$ to accuracy of $\varepsilon = 10^{-11}$ with initial point $p_0 = 1$.

Answer. $p_1 = g(p_0) = g(1) = \ln(3) = 1.0986$.

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0| \leqq \varepsilon,$$

$$\frac{\left(\frac{2}{3}\right)^n}{1 - \frac{2}{3}}|1.0986 - 1| \leqq 10^{-11} \Rightarrow \left(\frac{2}{3}\right)^n (0.2958) \leqq 10^{-11} \Rightarrow$$

$$\left(\frac{2}{3}\right)^n \leqq 3.3807 \times 10^{-11} \Rightarrow \log\left(\frac{2}{3}\right)^n \leqq \log(3.3807 \times 10^{-11}) \Rightarrow$$

$$n \log\left(\frac{2}{3}\right) \leqq \log(3.3807 \times 10^{-11}) \Rightarrow n \geqq \frac{\log(3.3807 \times 10^{-11})}{\log\left(\frac{2}{3}\right)} = 59.463.$$

Therefore, we deduce that $n \geqq 60$.