İ s t a n b u l K ü l t ü r U n i v e r s i t y
Department of Computer Engineering

MAT 002 - Numerical Methods
Fall 2011-2012

Second Midterm

SOLUTIONS

May 3, 2012

Directions – You have 110 minutes to complete the exam. Please do not leave the examination room in the first 30 minutes of the exam. There are six questions, of varying credit (100 points total). Indicate clearly your final answer to each question. You are allowed to use a calculator. During the exam, please turn off your cell phone(s). You cannot use the book or your notes. You have one page for "cheat-sheet" notes at the end of the exam papers. Do use the **radian mode** on your calculator when using the trigonometry buttons. Please use **five-decimal digit** in your calculations. The answer key to this exam will be posted on Department of Mathematics and Computer Science board after the exam.

Good luck! *Emel Yavuz Duman, PhD.*

Question 1.	
Question 2.	
Question 3.	

Question 4.	
Question 5.	
Question 6.	

Total

Question 1. 14 *points*

Consider the following table:

x_i	10	15	17
$f(x_i)$	35	10	14

Using the Second Lagrange Interpolating Polynomial, find the approximated solution of the equation $f(x) = 11$.

Answer. Since

$$\begin{aligned} f(x) &\approx P_2(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x) \\ &\approx 35\frac{(x-15)(x-17)}{(10-15)(10-17)} + 10\frac{(x-10)(x-17)}{(15-10)(15-17)} + 14\frac{(x-10)(x-15)}{(17-10)(17-15)} \\ &\approx (x-15)(x-17) - (x-10)(x-17) + (x-10)(x-15) \\ &\approx x^2 - 32x + 255 - x^2 + 27x - 170 + x^2 - 25x + 150 \\ &\approx x^2 - 30x + 235, \end{aligned}$$

then we have

$$f(x) \approx x^2 - 30x + 235.$$

Therefore approximated solutions of the equation $f(x) = 11$ are

$$f(x) = 11 \approx x^2 - 30x + 235 \Rightarrow x^2 - 30x + 224 = 0 \Rightarrow x_1 = 14 \text{ and } x_2 = 16.$$

Question 2. $12 + 3$ *points*

Use the Newton's Forward Difference Formula to approximate $\sqrt{5}$ with the function $f(x) = 5^x$ and the values $x_0 = -1$, $x_1 = 0$, $x_2 = 1$ and $x_3 = 2$. Also, compute the absolute error in this approximation

Answer.

i	x_i	$f(x_i)$	$\Delta f(x_i)$	$\Delta^2 f(x_i)$	$\Delta^3 f(x_i)$
0	-1	$0.2 = f(x_0)$	$0.8 = \Delta f(x_0)$	$3.2 = \Delta^2 f(x_0)$	$12.8 = \Delta^3 f(x_0)$
1	0	1	4	16	
2	1	5	20		
3	2	25			

$$x = 1/2 = 0.5 \Rightarrow s = \frac{x - x_0}{h} = \frac{0.5-(-1)}{1} = 1.5,$$

$$\begin{aligned} f(0.5) \approx P_3(0.5) &= f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \frac{s(s-1)(s-2)}{3!}\Delta^3 f(x_0) \\ &= 0.2 + (1.5)0.8 + \frac{(1.5)(0.5)}{2}3.2 + \frac{(1.5)(0.5)(-0.5)}{6}12.8 \\ &= 1.8 \end{aligned}$$

$$f(0.5) = 5^{0.5} = \sqrt{5} = 2.2361 \Rightarrow \text{Absolute Error} :|2.2361 - 1.8| = 0.4361.$$

Question 3. $(4+4)+7$ *points*

(a) Approximate the integral

$$\int_1^{1.5} x^2 \ln x dx$$

using the Trapezoidal and Simpson's rules.

Answer.

Trapezoidal Rule: $f(x) = x^2 \ln x$, $a = x_0 = 1$, $b = x_1 = 1.5$, $h = b - a = 1.5 - 1 = 0.5$

$$\begin{aligned}\int_a^b f(x)dx = \int_1^{1.5} x^2 \ln x dx &\approx \frac{h}{2}(f(x_0) + f(x_1)) \\ &\approx \frac{0.5}{2}(f(1) + f(1.5)) \\ &\approx \frac{0.5}{2}(1^2(\ln 1) + 1.5^2(\ln 1.5)) \\ &\approx 0.22807.\end{aligned}$$

Simpson's Rule: $f(x) = x^2 \ln x$, $a = x_0 = 1$, $b = x_2 = 1.5$, $h = \frac{b-a}{2} = \frac{1.5-1}{2} = 0.25$, $x_1 = x_0 + h = 1 + 0.25 = 1.25$

$$\begin{aligned}\int_a^b f(x)dx = \int_1^{1.5} x^2 \ln x dx &\approx \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2)) \\ &\approx \frac{0.25}{3}(f(1) + 4f(1.25) + f(1.5)) \\ &\approx \frac{0.25}{3}(1^2(\ln 1) + 4(1.25^2)(\ln 1.25) + (1.5^2)(\ln 1.5)) \\ &\approx 0.19225.\end{aligned}$$

(b) Find a bound for the error in the Trapezoidal rule approximation.

Answer. Since

$$f(x) = x^2 \ln x \Rightarrow f'(x) = 2x \ln x + x^2 \frac{1}{x} = 2x \ln x + x = x(2 \ln x + 1) \Rightarrow$$

$$f''(x) = 2 \ln x + 1 + x\frac{2}{x} = 2 \ln x + 1 + 2 = 2 \ln x + 3,$$

then we have

$$E = \left|\frac{h^3}{12} f''(z)\right| \leqq \max_{1 \leqq z \leqq 1.5} \left|\frac{0.5^3}{12}(2 \ln z + 3)\right| = \left|\frac{0.5^3}{12}(2 \ln(1.5) + 3)\right| = 0.039697.$$

Question 4. *7 + 7 points*

Approximate the integral

$$\int_0^3 e^{x^2} \tan x dx$$

(a) using the midpoint rule

Answer. Since $h = \frac{b-a}{2} = \frac{3-0}{2} = 1.5$, then we have

$$a = x_{-1} = 0, x_0 = x_{-1} + h = 0 + 1.5 = 1.5 \text{ and } x_1 = b = 3.$$

Therefore

$$\int_0^3 e^{x^2} \tan x dx \approx 2hf(x_0) = 2(1.5)\left(e^{1.5^2} \tan(1.5)\right) = 401.37.$$

(b) using Simpson's 3/8 rule

Answer. Since $h = \frac{b-a}{3} = \frac{3-0}{3} = 1$, then we have

$$a = x_0 = 0, x_1 = x_0 + h = 0 + 1 = 1, x_2 = x_0 + 2h = 0 + 2(1) = 2 \text{ and } x_3 = b = 3.$$

Therefore

$$\begin{aligned}\int_0^3 e^{x^2} \tan x dx &\approx \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)] \\ &\approx \frac{3 \cdot 1}{8}\left(e^{0^2} \tan(0) + 3e^{1^2} \tan(1) + 3e^{2^2} \tan(2) + e^{3^2} \tan(3)\right) \\ &\approx -562.60.\end{aligned}$$

Question 5. *7 + 7 points*

Neville's method is used to approximate $f(0.5)$ as follows. Complete the table.

i	x_i	$Q_{i,0}$	$Q_{i,1}$	$Q_{i,2}$
0	0	$0 = Q_{0,0}$		
1	0.4	$2.8 = Q_{1,0}$	$3.5 = Q_{1,1}$	
2	0.7	$a = Q_{2,0}$	$b = Q_{2,1}$	$27/7 = Q_{2,2}$

Answer.

$$\begin{aligned}Q_{2,2}(0.5) = \frac{27}{7} &= \frac{(0.5 - x_0)Q_{2,1} - (0.5 - x_2)Q_{1,1}}{x_2 - x_0} \\ \frac{27}{7} &= \frac{(0.5 - 0)b - (0.5 - 0.7)3.5}{0.7 - 0} = \frac{0.5b + 0.7}{0.7} \Rightarrow \\ \frac{27}{7} &= \frac{5b + 7}{7} \Rightarrow b = 4,\end{aligned}$$

$$\begin{aligned}Q_{2,1}(0.5) = 4 &= \frac{(0.5 - x_1)Q_{2,0} - (0.5 - x_2)Q_{1,0}}{x_2 - x_1} \\ 4 &= \frac{(0.5 - 0.4)a - (0.5 - 0.7)2.8}{0.7 - 0.4} = \frac{0.1a + 0.56}{0.3} \Rightarrow \\ 4 &= \frac{10a + 56}{30} \Rightarrow a = 6.4.\end{aligned}$$

Question 6. $10 + 4$ *points*

Let we have the following table:

x_i	-5	7	11	18
$f(x_i)$	-195	273	1261	5762

(a) Use Newton's divided difference method to obtain the third order interpolating polynomial $P_3(x)$.

Answer. Since

i	x_i	$f[x_i]$	$f[x_{i-1}, x_i]$	$f[x_{i-2}, x_{i-1}, x_i]$	$f[x_{i-3}, x_{i-2}, x_{i-1}, x_i]$
0	-5	$-195 = a_0$			
			$39 = a_1$		
1	7	273		$13 = a_2$	
			247		$1 = a_3$
2	11	1261		36	
			643		
3	18	5762			

thus

$$\begin{aligned} P_3(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) \\ &= -195 + 39(x + 5) + 13(x + 5)(x - 7) + 1(x + 5)(x - 7)(x - 11) \\ &= x^3 - 70. \end{aligned}$$

(b) Use $P_3(x)$ in (a) to approximate $f(0)$.

Answer.

$$f(0) \approx P_3(0) = 0^3 - 70 = -70.$$

Question 7. *7 + 7 points*

Suppose you are given the data in the following table:

x_i	-0.10	0.05	0.10	0.20	0.25	0.35	0.40	0.50	0.65
$f(x_i)$	29.91	30.053	30.11	30.24	30.323	30.473	30.56	30.75	31.073

(a) Find the *best approximation value* for $f'(0.5)$ using the *three point formula.*
Answer. We use the three point midpoint formula with $h = 0.15$

$$\begin{aligned} f'(0.5) &\approx \frac{1}{2h}[f(0.5+0.15) - f(0.5-0.15)] \\ &\approx \frac{1}{2(0.15)}[f(0.65) - f(0.35)] \\ &\approx \frac{1}{2(0.15)}[31.073 - 30.473] \\ &\approx 2. \end{aligned}$$

(b) Find the *best approximation value* for $f'(0.5)$ using the *five point formula.*
Answer. We use the five point endpoint formula with $h = -0.15$

$$\begin{aligned} f'(0.5) &\approx \frac{1}{12h}[25f(0.50) - 48f(0.50-0.15) + 36f(0.50-0.30) \\ &\qquad - 16f(0.50-0.45) + 3f(0.50-0.60)] \\ &\approx \frac{1}{12(0.15)}[25f(0.50) - 48f(0.35) + 36f(0.20) - 16f(0.05) + 3f(-0.10)] \\ &\approx \frac{1}{12(0.15)}[25(30.75) - 48(30.473) + 36(30.24) - 16(30.053) + 3(29.91)] \\ &\approx 1.9822. \end{aligned}$$