Lecture 1: Introduction

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MCB1007 Introduction to Probability and Statistics
İstanbul Kültür University
Outline

1. Sets
2. Combinatorial Methods
3. Binomial Coefficients
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Office Hours:  Friday 09:00 - 12:00, the AK / 3-A-03/05

Grading:  Two midterms:  30% + 30%; final: 40%.  A student who scores less than 15 will receive F grade.
Exam Dates:  Will be announced
1. Sets

2. Combinatorial Methods

3. Binomial Coefficients
Outline

1. Sets
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3. Binomial Coefficients
At the foundations of probability and statistics, and of mathematics in general, is the concept of a set.

**Definition 1**

A set can be thought of as a collection of objects, called *members* or *elements* of the set.

In general, unless otherwise specified, we denote a set by a capital letter such as $A, B, C$, and an element by a lower case letter such as $a, b, c$. If an element $a$ belongs to a set $C$ we write $a \in C$. If $a$ does not belong to $C$ we write $a \notin C$. If both $a$ and $b$ belong to $C$ we write $a, b \in C$. In order for a set to be well-defined, as we shall always assume, we must be able to determine whether a particular object does or does not belong to the set.
A set can be defined by actually listing its elements or, if this is not possible, by describing some property held by all members and by no nonmembers. The first is called the *roster method* and the second is called the *property method*.

**Example 2**

The set of all vowels in the English alphabet can be defined by the roster method as \{a, e, i, o, u\} or by the property method as \{x | x is a vowel\}, read “the set of all elements \(x\) such that \(x\) is a vowel” where the vertical line \(\mid\) is read “such that” or “given that”.

**Example 3**

The set \{x | x is a triangle in a plane\}, is the set of all triangles in a plane. Note that the roster method cannot be used here.

**Example 4**

If we toss a pair of ordinary dice the possible “numbers” or “points” which can come up on the uppermost face of each die are elements of the set \{1, 2, 3, 4, 5, 6\}. 
The Concept of a Set

- If each element of a set $A$ also belongs to a set $B$ we call $A$ a *subset* of $B$, written $A \subset B$ or $B \supset A$ and read “$A$ is contained in $B$” or “$B$ contains $A$” respectively. It follows that for all sets $A$ we have $A \subset A$.

- If $A \subset B$ and $B \subset A$ we call $A$ and $B$ *equal* and write $A = B$. In such case $A$ and $B$ have exactly the same elements.

- If $A$ is not equal to $B$, i.e. if $A$ and $B$ do not have exactly the same elements, we write $A \neq B$.

- If $A \subset B$ but $A \neq B$ we call $A$ a *proper subset* of $B$. 
Example 5

\{a, i, u\} is a proper subset of \{a, e, i, o, u\}.

Example 6

\{i, o, a, u, e\} is a subset, but not a proper subset, of \{a, e, i, o, u\}, because the two sets are equal. Note that a mere rearrangement of elements does not change the set.

Example 7

In tossing a die the possible outcomes where the die comes up “even” are elements of the set \{2, 4, 6\}, which is a (proper) subset of the set of all possible outcomes \{1, 2, 3, 4, 5, 6\}.
The following theorem is true for any sets $A, B, C$.

**Theorem 8**

*If $A \subset B$ and $B \subset C$, then $A \subset C$.***

**Proof.**

We must show that every element in $A$ is in $C$. To this end we note that if $x \in A$, then $x \in B$ (because $A \subset B$) and therefore $x \in C$ (because $B \subset C$). Hence $A \subset C$. □
A *universal set* or *space* is a set which contains all objects, including itself and denoted by $\mathcal{U}$. The elements of a space are often called *points* of the space.

It is useful to consider a set having no elements at all. This is called the *empty set* or *null set* and is denoted by $\emptyset$. It is a subset of any set.
Example 9

An important set already familiar is the set \( \mathbb{R} \) of real numbers such as 3, \(-2\), \(\sqrt{2}\), \(\pi\), which can be represented by points on a real line such as the \(x\)-axis. If \(a\) and \(b\) are real numbers such that \(a < b\), the subsets \(\{x|a \leq x \leq b\}\) and \(\{x|a < x < b\}\) of \(\mathbb{R}\) (often denoted briefly by \(a \leq x \leq b\) and \(a < x < b\)) are called closed and open intervals respectively. Subsets such as \(\{x|a \leq x < b\}\) or \(\{x|a < x \leq b\}\) are then called half open or half closed intervals.

Example 10

The set of all real numbers \(x\) such that \(x^2 = -1\), written \(\{x|x^2 = -1\}\), is the null or empty set since there are no real numbers whose squares are equal to \(-1\). If we include complex numbers, however, the set is not empty.

Example 11

If we toss a die the set of all possible outcomes is the universe \(\{1, 2, 3, 4, 5, 6\}\). The set of outcomes consisting of faces 7 or 11 on a single die is the null set.
Venn Diagrams

A universe \( U \) can be represented geometrically by the set of points inside a rectangle. In such case subsets of \( U \) (such as \( A \) and \( B \) shown in the figure) are represented by sets of points inside circles. Such diagrams, called \textit{Venn diagrams}, often serve to provide geometric intuition regarding possible relationships between sets.

\[ \text{Figure 1: Venn Diagrams} \]
1 Union. The set of all elements (or points) which belong to either $A$ or $B$ or both $A$ and $B$ is called the union of $A$ and $B$ and is denoted by $A \cup B$ (shaded in the figure).

Figure 2 : $A \cup B$
**Intersection.** The set of all elements which belong to both $A$ and $B$ is called the intersection of $A$ and $B$ and is denoted by $A \cap B$ (shaded in the figure).

Two sets $A$ and $B$ such that $A \cap B = \emptyset$, i.e. which have no elements in common, are called *disjoint* sets.
**Difference.** The set consisting of all elements of $A$ which do not belong to $B$ is called the difference of $A$ and $B$, denoted by $A - B$ (shaded in the figure).

Figure 4: $A - B$
Complement. If \( B \subset A \) then \( A - B \) is called the complement of \( B \) relative to \( A \) and is denoted by \( B'_A \) (shaded on the left in the figure). If \( A = \mathcal{U} \), the universal set, we refer to \( \mathcal{U} - B \) as simply the complement of \( B \) and denote it by \( B' \) (shaded on the right in the figure). The complement of \( A \cup B \) is denoted by \( (A \cup B)' \).

\[ \]

Figure 5 : \( B'_A \) and \( B' \)
Some Theorems Involving Sets

Theorem 12 (Commutative law for unions)

$$A \cup B = B \cup A.$$  

Theorem 13 (Associative law for unions)

$$A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C.$$  

Theorem 14 (Commutative law for intersections)

$$A \cap B = B \cap A.$$  

Theorem 15 (Associative law for intersections)

$$A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C.$$
Theorem 16 (First distributive law)

\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \]

Theorem 17 (Second distributive law)

\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C). \]

Theorem 18

\[ A - B = A \cap B'. \]

Proof.

\[ A - B = \{ x | x \in A \text{ and } x \notin B \} \]
\[ = \{ x | x \in A \text{ and } x \in B' \} \]
\[ = A \cap B'. \]
Theorem 19
If $A \subset B$, then $A' \supset B'$ or $B' \subset A'$.

Theorem 20
$A \cup \emptyset = A$, $A \cap \emptyset = \emptyset$.

Theorem 21
$A \cup U = U$, $A \cap U = A$. 
Theorem 22 (De Morgan’s laws)

1. \((A \cup B)' = A' \cap B'\),
2. \((A \cap B)' = A' \cup B'\).

Proof.

\[(A \cup B)' = \{x | x \notin A \cup B\}\]
\[= \{x | x \notin A \text{ and } x \notin B\}\]
\[= \{x | x \in A' \text{ and } x \in B'\}\]
\[= A' \cap B'.\]

\[(A \cap B)' = \{x | x \notin A \cap B\}\]
\[= \{x | x \notin A \text{ or } x \notin B\}\]
\[= \{x | x \in A' \text{ or } x \in B'\}\]
\[= A' \cup B'.\]
Theorem 23

\[ A = (A \cap B) \cup (A \cap B'). \]

Proof.

\[ A = \{x | x \in A\} \]
\[ = \{x | x \in A \cap B \text{ or } x \in A \cap B'\} \]
\[ = (A \cap B) \cup (A \cap B'). \]
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1. Sets
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Fundamental Principle of Counting

**Theorem 24**

*If an operation consists of $k$ steps, of which the first can be done in $n_1$ ways, for each of these the second step can be done in $n_2$ ways, for each of the first two the third step can be done in $n_3$ ways, an so forth, then the whole operation can be done in $n_1 \cdot n_2 \cdots n_k$ ways.*
Example 25

There are three shirts of different colors, three pairs of pants of different styles and two jackets in the closet. How many ways can you dress yourself with one shirt, one jacket and one pair of pants selected from the closet?

Solution. \( k = 3; \ n_1 = 3, \ n_2 = 3, \ n_3 = 2 \). There are \( n_1 \cdot n_2 \cdot n_3 = 3 \cdot 3 \cdot 2 = 18 \) (possible) ways that you can dress yourself.

Figure 6: Tree Diagram
Example 26
How many possible outcomes are there when we roll a pair of dice, one red and one green?

Solution. The red die can land in any one of six ways, and for each of these six ways the green die can also land in six ways. Therefore, the pair of dice can land in

$$6 \cdot 6 = 36$$

ways.

Example 27
A child has 3 pockets and 4 coins. In how many ways can he put the coins in his pocket?

Solution. First coin can be put in 3 ways, similarly second, third and forth coins also can be put in 3 ways. So the total number of ways is

$$3 \cdot 3 \cdot 3 \cdot 3 = 3^4 = 81.$$
Permutation

Definition 28

A *permutation* is an arrangement of objects in specific order.

The order of the arrangement is important!!!
Example 29
Consider, four students walking toward their school entrance. How many different ways could they arrange themselves in this side-by-side pattern?

Solution.

1,2,3,4  2,1,3,4  3,2,1,4  4,2,3,1
1,2,4,3  2,1,4,3  3,2,4,1  4,2,1,3
1,3,2,4  2,3,1,4  3,1,2,4  4,3,2,1
1,3,4,2  2,3,4,1  3,1,4,2  4,3,1,2
1,4,2,3  2,4,1,3  3,4,2,1  4,1,2,3
1,4,3,2  2,4,3,1  3,4,1,2  4,1,3,2

We observe that there are 24 different arrangements, or permutations, of the four students walking side-by-side by listing all possible arrangements. Since there are four choices to select a student for the first position, then three for the second position, and then two for the third position, leaving only one letter for the fourth position the total number of permutations is

\[4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24\]
Generalizing the argument used in the preceding example, we find that $n$ distinct objects can be arranged in

$$n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1 = n!$$

different ways.

By definition we let $0! = 1$.

**Theorem 30**

The number of permutation of $n$ distinct object is $n!$.

**Example 31**

In how many different ways can the five starting players of a basketball team be introduced to the public?

**Solution.** There are $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways in which they can be introduced.
Example 32

The number of permutations of the four number 1, 2, 3 and 4 is 24, but what is the number of permutations if we take only two of the four numbers or, as usually put, if we take the four numbers two at a time?

Solution. We have two positions to fill, with four choices for the first and then three choices for the second. Therefore, the number of permutations is

$$4 \cdot 3 = 12.$$
Generalizing the argument that we used in the preceding example, we find that \( n \) district objects taken \( r \) at a time, for \( r > 0 \), can be arranged in

\[
nP_r = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)
\]

ways. Therefore, we can write

**Theorem 33**

*The number of permutation of \( n \) distinct objects taken \( r \) at a time is*

\[
nP_r = \frac{n!}{(n-r)!}
\]

for \( r = 0, 1, 2, \ldots, n \).

\[
nP_n = \frac{n!}{(n-n)!} = \frac{n!}{1} = n!, \quad nP_1 = \frac{n!}{(n-1)!} = n, \quad nP_0 = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1
\]
Example 34

The number of different arrangements, or permutations, consisting of 3 letters each that can be formed from the 7 letters $A, B, C, D, E, F, G$ is

$$7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4!} = 7 \cdot 6 \cdot 5 = 210.$$ 

Example 35

Four names are drawn from among the 24 members of a club for the offices of president, vice president, treasurer, and secretary. In how many different ways can this be done?

Solution.

$$24P_4 = \frac{24!}{(24-4)!} = \frac{24!}{20!} = \frac{24 \cdot 23 \cdot 22 \cdot 21 \cdot 20!}{20!} = 24 \cdot 23 \cdot 22 \cdot 21 = 255,024.$$
Theorem 36

The number of permutations of $n$ objects of which $n_1$ are of one kind, $n_2$ are of a second kind, \ldots, $n_k$ are of a $k^{th}$ type, and $n = n_1 + n_2 + \cdots + n_k$ is

$$nP_{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! \cdot n_2! \cdots n_k!}.$$  

Example 37

The number of different permutations of the 11 letters of the word MISSISSIPPI, which consists of 1 M, 4 I’s, 4 S’s, and 2 P’s, is

$$11P_{1,4,4,2} = \frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!} = 34,650.$$
Example 38

In how many different ways can three copies of one novel and one copy each of four other novels be arranged on a shelf?

Solution. \(7P_{3,1,1,1,1} = \frac{7!}{3! \cdot 1! \cdot 1! \cdot 1!} = 7 \cdot 6 \cdot 5 \cdot 4 = 840.\)

Example 39

In how many ways can two paintings by Monet, three paintings by Renoir, and two paintings by Degas be hung side by side on a museum wall if we do not distinguish between the paintings by the same artists?

Solution. \(7P_{2,3,2} = \frac{7!}{2! \cdot 3! \cdot 2!} = 210.\)
The number of permutations of $n$ distinct objects arranged in a circle is $(n-1)!$.

Example 41

How many circular permutations are there of four persons playing bridge?

Solution. If we arbitrarily consider the position of one of the four players as fixed, we can seat (arrange) the other three players in $3! = 6$ different ways. In other words, there are six different circular permutations.

Figure 7: Circular Permutations
Example 42

Seven men and seven women have to sit around a circular table so that no 2 women are together. In how many different ways can this be done?

Solution. 7 men in a circle can be arranged in \((7 - 1)! = 6!\) ways and if we place 7 women in empty slots between them then no two women will be together. The number of arrangement of these 7 women will be 7! and not 6! because if we shift them by one position we will get different arrangement because of the neighboring men. So the answer is indeed

\[ 6! \cdot 7! = 3,628,800. \]
Combinations

Definition 43

A combination is the number of ways of picking $r$ unordered outcomes from $n$ possibilities.

The order of the arrangement is not important!!!

ABCD is a different permutation from BADC, but ABCD and BADC are the same combinations.
Example 44

In how many different ways can a person gathering data for a market research organization select three of the 20 households living in a certain apartment complex?

Solution. If we care about the order in which the households are selected, the answer is

\[ 20P_3 = \frac{20!}{(20 - 3)!} = 20 \cdot 19 \cdot 18 = 6,840 \]

but each set of three households would then be counted \(3! = 6\) times. If we do not care about the order in which the households are selected, there are only

\[ \frac{6,840}{6} = 1,140 \]

ways in which the person gathering the data can do his or her job.
Theorem 45

The number of combinations of \( n \) distinct objects taken \( r \) at a time is

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!}
\]

for all \( r = 0, 1, 2, \ldots, n \).

Example 46

How many ways can a committee be formed by selecting 3 people from a group of 10 candidates?

Solution. \( \binom{10}{3} = \frac{10!}{3! \cdot 7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{3! \cdot 7!} = 120 \).
Figure 8: Standard 52-Card Deck

Figure 9: Tail and Head
Example 47

In how many different ways can six tosses of a coin yield two heads and four tails?

Solution. \( \binom{6}{2} = \binom{6}{4} = \frac{6!}{2! \cdot 4!} = 15. \)

Example 48

How many different 5-card hands include 4 of a kind and one other card?

Solution. \( \binom{4}{4} \cdot \binom{48}{1} \cdot 13 = \frac{4!}{4! \cdot 0!} \cdot \frac{48!}{1! \cdot 47!} \cdot 13 = 1 \cdot 48 \cdot 13 = 624. \)

Example 49

How many ways are there to deal a five-card hand consisting of three eight’s and two sevens.

Solution. \( \binom{4}{3} \cdot \binom{4}{2} = \frac{4!}{3! \cdot 1!} \cdot \frac{4!}{2! \cdot 2!} = 4 \cdot 6 = 24. \)
Example 50

How many different committees of 2 chemists and 1 physicist can be formed from the 4 chemists and 3 physicists on the faculty of a small college?

Solution. \( \binom{4}{2} \cdot \binom{3}{1} = \frac{4!}{2! \cdot 2!} \cdot \frac{3!}{1! \cdot 2!} = 6 \cdot 3 = 18 \).

Example 51

From a group of 7 men and 6 women, five persons are to be selected to form a committee so that at least 3 men are there on the committee. In how many ways can it be done?

Solution. We may have (3 men and 2 women) or (4 men and 1 woman) or (5 men only).

\[
\binom{7}{3} \cdot \binom{6}{2} + \binom{7}{4} \cdot \binom{6}{1} + \binom{7}{5} \cdot \binom{6}{0} = 525 + 210 + 21 = 756.
\]
Outline

1. Sets
2. Combinatorial Methods
3. Binomial Coefficients
Binomial Coefficients

\[(x + y)^0 = 1\]
\[(x + y)^1 = 1x + 1y\]
\[(x + y)^2 = 1x^2 + 2xy + 1y^2\]
\[(x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3\]
\[(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4\]
\[(x + y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5\]

\[\vdots\]
Definition 52

Pascal’s triangle is a triangular array of the binomial coefficients in a triangle.

Figure 10: Pascal’s Triangle
Theorem 53

\[(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r\]

\[= \binom{n}{0} x^{n-0} y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \ldots \]
\[+ \binom{n}{n-1} x^{n-(n-1)} y^{n-1} + \binom{n}{n} x^{n-n} y^n\]

\[= x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots \]
\[+ \binom{n}{n-1} xy^{n-1} + y^n\]

for any positive integer \(n\).
\[(x + y)^2 = \binom{2}{0} x^2 y^0 + \binom{2}{1} xy + \binom{2}{2} y^2 x^0\]

\[(x + y)^3 = \binom{3}{0} x^3 y^0 + \binom{3}{1} x^2 y + \binom{3}{2} xy^2 + \binom{3}{3} x^0 y^3\]

\[(x + y)^4 = \binom{4}{0} x^4 y^0 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} xy^3 + \binom{4}{4} x^0 y^4\]

\[\vdots\]
Example 54

What is the coefficient of $x^{10}$ in the expansion of $(x + 1)^{12}$?

Solution.

$$(x + 1)^{12} = \sum_{r=0}^{12} \binom{12}{r} x^{12-r} 1^r \Rightarrow x^{10} = x^{12-r} \Rightarrow r = 2.$$  

$$\binom{12}{2} x^{12-2} 1^2 = \frac{12!}{2! \cdot 10!} x^{10} = \frac{12 \cdot 11}{2} x^{10} = 66x^{10}.$$  

The coefficient of $x^{10}$ in $(x + 1)^{12}$ is 66.
Example 55

What is the coefficient of $x$ in the expansion of $(2x - 1)^8$.

Solution.

$$(2x - 1)^8 = \sum_{r=0}^{8} \binom{8}{r} (2x)^{8-r}(-1)^r \Rightarrow x^1 = x^{8-r} \Rightarrow r = 7.$$

$$\binom{8}{7}(2x)^{8-7}(-1)^7 = \frac{8!}{7! \cdot 1!}(2x)(-1) = -16x.$$

The coefficient of $x^1$ in $(2x - 1)^8$ is $-16$. 
Example 56

Find the constant term in the expansion of \((x^2 - \frac{2}{x})^6\).

Solution.

\[
(x^2 - \frac{2}{x})^6 = \sum_{r=0}^{6} \binom{6}{r} (x^2)^{6-r} \left(-\frac{2}{x}\right)^r
\]

\[
= \sum_{r=0}^{6} \binom{6}{r} x^{12-2r} x^{-r} (-2)^r
\]

\[
= \sum_{r=0}^{6} \binom{6}{r} x^{12-3r} (-2)^r \Rightarrow
\]

\[x^0 = x^{12-3r} \Rightarrow r = 4.\]

\[
\binom{6}{4} x^{12-3\cdot4} (-2)^4 = \frac{6!}{4! \cdot 2!} \cdot 2^4 = 240.
\]

The constant term in \((x^2 - \frac{2}{x})^6\) is 240.
Thank You!!!