Final Exam

January 11, 2011

Directions — You have 90 minutes to complete the exam. Please do not leave the examination room in the first 30 minutes of the exam. There are eight questions, of varying credit (100 points total). You must show your working to get full marks for a question, an answer alone will not earn credit. Indicate clearly your final answer to each question. You are allowed to use a calculator. During the exam, please turn off your cell phone(s). You cannot use the book or your notes. You have two double-sided pages for “cheat-sheet” notes and some tables. You can keep them for you end of the exam. The answer key to this exam will be posted on Department of Mathematics and Computer Science board after the exam.

Good luck! Emel Yavuz Duman, PhD.

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Mark ________
The probability density function of $X$ is given by

$$f(x) = \begin{cases} 
  a + bx^2, & 0 \leq x \leq 1, \\
  0, & \text{otherwise.}
\end{cases}$$

If $E[X] = \frac{3}{5}$, find $a$ and $b$.

**Answer.** Since $E[X] = \int_{-\infty}^{\infty} xf(x)dx = \frac{3}{5}$ and $\int_{-\infty}^{\infty} f(x)dx = 1$, thus

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_{0}^{1} x(a + bx^2)dx = \left(\frac{a}{2}x^2 + \frac{b}{4}x^4\right)_0^1 = \frac{a}{2} + \frac{b}{4} = \frac{3}{5}$$

and

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{1} (a + bx^2)dx = \left(ax + \frac{b}{3}x^3\right)_0^1 = a + \frac{b}{3} = 1.$$ 

So we have

$$\frac{a}{2} + \frac{b}{4} = \frac{3}{5} \Rightarrow a = \frac{3}{5}, \ b = \frac{6}{5},$$

$$a + \frac{b}{3} = 1.$$ 

**Question 2.**

On a multiple-choice exam with 3 possible answers for each of the 5 questions, what is the probability that a student would get 4 or more correct answers just by guessing? *Hint: Use the Binomial distribution.*

**Answer.** Since $n = 5$, $x = 4$, $p = \frac{1}{3}$, $q = \frac{2}{3}$, then

$$P(X \geq 4) = P(X = 4) + P(X = 5) = \binom{5}{4} \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^1 + \binom{5}{5} \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^0$$

$$= 5 \cdot \frac{1}{3}^4 \cdot \frac{1}{3}^1 + \frac{1}{3}^5 \cdot (10 + 1) = \frac{11}{243} \cong 0.045.$$ 

**Question 3.**

You select a random sample of $n = 1600$ tires from an ongoing production process in which 8% of all such tires produced are defective. What is the probability that 150 or fewer tires will be defective? *Hint: Use the normal distribution to approximate the binomial.*

**Answer.** Since $n = 1600$, $p = 0.08$, $q = 0.92$, then $\mu = np = (1600)(0.08) = 128$ and $\sigma = \sqrt{npq} = \sqrt{(1.600)(0.08)(0.92)} \cong 10.85$. We require the probability that the number of defective ones will be lie between 0 and 150 or considering the data as continuous, between $-0.5$ and $150.5$.

$$-0.5 \text{ in standard units} = \frac{-0.5 - 128}{10.85} \cong -11.84$$

and

$$150.5 \text{ in standard units} = \frac{150.5 - 128}{10.85} \cong 2.07.$$ 

Required probability $= \text{(area under normal curve between } z = -11.84 \text{ and } z = 2.07)$

$= 0.5 + 0.4808 = 0.9808.$
Question 4.  10 points
A certain nucleus contains $1 \times 10^6$ genes. Each gene has probability $3, 2 \times 10^{-6}$ of being mutated. What is the probability that the nucleus contains at most 2 mutated genes? 

**Hint:** Use the Poisson distribution to approximate the binomial.

**Answer.** Since $n = 1 \times 10^6$, $p = 3, 2 \times 10^{-6}$, then $\mu = np = (1 \times 10^6)(3, 2 \times 10^{-6}) = 3, 2$.

\[
P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = e^{-3, 2} + 3, 2e^{-3, 2} + \frac{(3, 2)^2}{2} e^{-3, 2} \\
= e^{-3, 2} \left( 1 + 3, 2 + \frac{(3, 2)^2}{2} \right) = e^{-3} e^{-0.2}(9, 32) = (0, 04979)(0, 8187)(9, 32) \\
\cong 0, 38.
\]

Question 5.  10 points

The joint density function of $X$ and $Y$ is given by

\[ f(x, y) = \begin{cases} \frac{1}{16}xy, & 0 \leq x \leq 2, 0 \leq y \leq 4, \\ 0, & \text{otherwise.} \end{cases} \]

Show that $Cov(X, Y) = \sigma_{XY} = 0$.

**Answer.** There are several ways to solve this problem. We show two of them: 

**Method 1.**

\[
E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x, y)dxdy = \frac{1}{16} \int_{0}^{4} \int_{0}^{2} x^2y^2dxdy \\
= \frac{1}{16} \int_{0}^{4} \frac{x^3}{3} y^2 \bigg|_{x=0}^{x=2} dy = \frac{1}{6} \int_{0}^{4} y^2dy = \frac{1}{6} \frac{y^3}{3} \bigg|_{y=0}^{y=4} = \frac{32}{9},
\]

\[
E(X) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xf(x, y)dxdy = \frac{1}{16} \int_{0}^{4} \int_{0}^{2} x^2ydx dy \\
= \frac{1}{16} \int_{0}^{4} \frac{x^3}{3} y \bigg|_{x=0}^{x=2} dy = \frac{1}{6} \int_{0}^{4} ydy = \frac{1}{6} \frac{y^2}{2} \bigg|_{y=0}^{y=4} = \frac{4}{3},
\]

\[
E(Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} yf(x, y)dxdy = \frac{1}{16} \int_{0}^{4} \int_{0}^{2} xy^2dx dy \\
= \frac{1}{16} \int_{0}^{4} \frac{x^2}{2} y^2 \bigg|_{x=0}^{x=2} dy = \frac{1}{8} \int_{0}^{4} y^2dy = \frac{1}{8} \frac{y^3}{3} \bigg|_{y=0}^{y=4} = \frac{8}{3},
\]

\[
Cov(X, Y) = \sigma_{XY} = E(XY) - E(X)E(Y) = \frac{32}{9} - \frac{4}{3} \cdot \frac{8}{3} = 0.
\]

**Method 2.** We know that if $X$ and $Y$ are independent random variables, then $Cov(X, Y) = 0$, and if $X$ and $Y$ are independent random variables, then $f(x, y) = f_1(x)f_2(y)$ where $f_1(x)$ and $f_2(y)$ are the marginal density functions of $X$ and $Y$, respectively. Since

\[
f_1(x) = \int_{v=-\infty}^{\infty} f(x, v)dv = \frac{1}{16} \int_{v=0}^{4} xvdv = \frac{1}{32} x^2 \bigg|_{v=0}^{v=4} = \frac{x}{2},
\]

and

\[
f_2(y) = \int_{u=-\infty}^{\infty} f(u, y)du = \frac{1}{16} \int_{u=0}^{2} uydud = \frac{1}{32} u^2y \bigg|_{u=0}^{u=2} = \frac{y}{8},
\]

thus $f_1(x)f_2(y) = \frac{x}{2} \cdot \frac{y}{8} = \frac{xy}{16} = f(x, y)$. 

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MAT 003, Fall 2010-2011, Final Exam 3  
Continued
A continuous random variable $X$ has the following probability density function

$$f(x) = \begin{cases} 4e^{-2x}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Find the moment generating function for $X$.

**Answer.**

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx = 4 \int_{0}^{\infty} e^{x(t-2)} dx = \frac{4}{t-2} e^{x(t-2)} \bigg|_{x=0}^{\infty} = \frac{4}{t-2} (0-1) = \frac{4}{2-t} \quad \text{for } t < 2$$

**Question 7.**

Given that $X$ has moment generating function

$$M_X(t) = \frac{1}{6} e^{-2t} + \frac{1}{3} e^{-t} + \frac{1}{4} e^t.$$  

(a) Find the first 3 moments about the origin.

**Answer.** Since $\mu'_r = \frac{d^r}{dt^r} M_X(t)|_{t=0}$, then

$$\frac{d}{dt} M_X(t) = -\frac{2}{6} e^{-2t} - \frac{1}{3} e^{-t} + \frac{1}{4} e^t \Rightarrow \frac{d}{dt} M_X(t)|_{t=0} = \frac{2}{6} - \frac{1}{3} + \frac{1}{4} = \frac{-5}{12} = \mu'_1 = \mu$$

$$\frac{d^2}{dt^2} M_X(t) = \frac{4}{6} e^{-2t} + \frac{1}{3} e^{-t} + \frac{1}{4} e^t \Rightarrow \frac{d^2}{dt^2} M_X(t)|_{t=0} = \frac{4}{6} + \frac{1}{3} + \frac{1}{4} = \frac{5}{4} = \mu'_2$$

$$\frac{d^3}{dt^3} M_X(t) = -\frac{8}{6} e^{-2t} - \frac{1}{3} e^{-t} + \frac{1}{4} e^t \Rightarrow \frac{d^3}{dt^3} M_X(t)|_{t=0} = \frac{-8}{6} - \frac{1}{3} + \frac{1}{4} = \frac{-17}{12} = \mu'_3$$

(b) Find the first 3 moments about the mean.

**Answer.**

$$\mu_1 = E[X - \mu] = E[X] - \mu = \mu - \mu = 0$$

$$\mu_2 = \mu'_2 - \mu^2 = \frac{5}{4} - \left( \frac{5}{12} \right)^2 = \frac{155}{144}$$

$$\mu_3 = \mu'_3 - 3\mu_2^2 + 2\mu^3 = -\frac{17}{12} - 3 \left( -\frac{5}{12} \right) \left( \frac{5}{4} \right) + 2 \left( -\frac{5}{12} \right)^3 = \frac{1}{864}.$$  

**Question 8.**

The joint probability function of two discrete random variables $X$ and $Y$ is given by

$$f(x, y) = \begin{cases} \frac{c}{12} (x + 2y), & 0 \leq x \leq 2, \ 0 \leq y \leq 1, \\ 0, & \text{otherwise}. \end{cases}$$

(a) Find the value of the constant $c$.

**Answer.**

<table>
<thead>
<tr>
<th>Y</th>
<th>$X$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$c/12$</td>
<td>2$c/12$</td>
<td>3$c/12$</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2$c/12$</td>
<td>3$c/12$</td>
<td>4$c/12$</td>
<td>9$c/12$</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>2$c/12$</td>
<td>4$c/12$</td>
<td>6$c/12$</td>
<td>12$c/12 = 1$ $\Rightarrow c = 1$</td>
<td></td>
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(b) Find $\text{Var}(X)$.

**Answer.**

$$
\mu_x = E(X) = \sum_x \sum_y x f(x, y) = \sum_x x \left( \sum_y f(x, y) \right) = 0 \cdot \frac{2}{12} + 1 \cdot \frac{4}{12} + 2 \cdot \frac{6}{12} = \frac{4}{3}
$$

$$
\text{Var}(X) = \sigma_X^2 = E[(X - \mu_x)^2] = \sum_x \sum_y (x - \mu_x)^2 f(x, y) = \sum_x \sum_y \left( x - \frac{4}{3} \right)^2 f(x, y)
$$

$$
= \left(0 - \frac{4}{3}\right)^2 \cdot \frac{2}{12} + \left(1 - \frac{4}{3}\right)^2 \cdot \frac{4}{12} + \left(2 - \frac{4}{3}\right)^2 \cdot \frac{6}{12} = \frac{15}{27}
$$

(c) Find $E(Y|X = 1)$.

**Answer.**

$$
f(y|x) = \frac{f(x, y)}{f_1(x)} \Rightarrow f(y|1) = \frac{f(1, y)}{f_1(1)} = \frac{1 + 2y}{\frac{12}{4}} = \frac{1 + 2y}{4}
$$

$$
E(Y|X = 1) = \sum_y y f(y|1) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{3}{4} = \frac{3}{4}
$$