MAT 003 - INTRODUCTION TO PROBABILITY THEORY AND STATISTICS
Fall 2011-2012

First Midterm

November 2, 2011

Number: Name:

Directions — You have 105 minutes to complete the exam. Please do not leave the examination room in the first half of the exam. There are seven questions, of varying credit (100 points total). You must show your working to get full marks for a question, an answer alone will not earn credit. Indicate clearly your final answer to each question. You are allowed to use a calculator. During the exam, please turn off your cell phone(s). You cannot use any book or your notes. You have one double-sided page for “cheat-sheet” notes at the end of the exam papers. The answer key to this exam will be posted on Department of Mathematics and Computer Science board after the exam. Good luck!

Emel Yavuz Duman, Ph.D.

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Suppose that there are three brands of a product which are equally popular in the sense that, for a randomly chosen consumer, the outcome of preference for one brand is as likely as that for any other. For 10 randomly selected consumers, the event $\mathcal{A}$ is defined as

$$A = \{4 \text{ persons prefer brand 1, 5 persons prefer brand 2, 1 person prefer brand 3}\}.$$ 

Find the probability of $A$.

**Answer.** Because there are 3 preferences for each person, the number of possible outcomes for ten persons is $3^{10}$. The assumption of equal popularity and random selection entail that all these outcomes are equally likely.

The number of elements in event $\mathcal{A}$ is precisely the number of partitions of 10 persons into three groups of sizes 4, 5 and 1. This count is obtained as

$$\frac{10!}{4!5!1!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 5! \cdot 1} = 10 \cdot 3 \cdot 7 \cdot 6 = 1260.$$ 

Therefore,

$$P(A) = \frac{\frac{10!}{4!5!1!}}{3^{10}} = \frac{1260}{59049} = \frac{140}{6561} \approx 0.02133821064.$$ 

**Question 2.**

Let $E$ and $F$ be two events. Prove Benferroni’s inequality, namely,

$$P(E \cap F) \geq P(E) + P(F) - 1.$$ 

**Answer.** For any two events we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$ 

As with all probabilities, we know that $0 \leq P(A \cup B) \leq 1$. Therefore

$$P(A) + P(B) - P(A \cap B) = P(A \cup B) \leq 1$$

and

$$P(A) + P(B) - P(A \cap B) \leq 1$$

which is equivalent to

$$P(E \cap F) \geq P(E) + P(F) - 1.$$
The Land of Nod lies in the monsoon zone, and has just two seasons, Wet and Dry. The Wet season lasts for 1/3 of the year, and the Dry season for 2/3 of the year. During the Wet season, the probability that it is raining is 3/4; during the Dry season, the probability that it is raining is 1/6.

(a) You visit the capital city, Oneirabad, on a random day of the year. What is the probability that it is raining when you arrive?

**Answer.** Let we define three events $W$, $D$ and $R$ as follows:

$W = \{\text{it is the wet season}\}$,

$D = \{\text{it is the dry season}\}$,

$R = \{\text{it is raining when you arrive}\}$.

We are given that

$$P(W) = \frac{1}{3}, \ P(D) = \frac{2}{3}, \ P(R|W) = \frac{3}{4}, \ P(R|D) = \frac{1}{6}.$$ 

Therefore, we have

$$P(R) = P(W)P(R|W) + P(D)P(R|D) = \left(\frac{1}{3}\right)\left(\frac{3}{4}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{6}\right) = \frac{13}{36} \cong 0.36\overline{1}.$$ 

(b) You visit Oneirabad on a random day, and it is raining when you arrive. Given this information, what is the probability that your visit is during the Wet season?

**Answer.** By Bayes’

$$P(W|R) = \frac{P(W \cap R)}{P(W \cap R) + P(D \cap R)} = \frac{P(W|P(R|W))}{P(W|P(R|W)) + P(D|P(R|D))}$$

$$= \frac{P(W)P(R|W)}{P(R)} = \frac{\left(\frac{1}{3}\right)\left(\frac{3}{4}\right)}{\frac{13}{36}} = \frac{9}{13} \cong 0.6923076923.$$ 

(c) You visit Oneirabad on a random day, and it is raining when you arrive. Given this information, what is the probability that it will be raining when you return to Oneirabad in a year’s time?

**Answer.** Let $R^*$ be the event given by

$R^* = \{\text{it is raining in a year’s time}\}$.

We want to find the value of

$$P(R^*|R) = \frac{P(R \cap R^*)}{P(R)}.$$ 

Since

$$P(R \cap R^*) = P(W)P(R \cap R^*|W) + P(D)P(R \cap R^*|D)$$

$$= \left(\frac{1}{3}\right)\left(\frac{3}{4}\right)^2 + \left(\frac{2}{3}\right)(\frac{1}{6})^2 = \frac{89}{432},$$

thus

$$P(R^*|R) = \frac{P(R \cap R^*)}{P(R)} = \frac{89/432}{13/36} = \frac{89}{156} \cong 0.5705128205.$$
It has been determined that the probability density function for the wait in line at a counter is given by,

\[
f(x) = \begin{cases} 
0 & \text{if } x < 0, \\
0.1e^{-\frac{x}{10}} & \text{if } x \geq 0,
\end{cases}
\]

where \( x \) is the number of minutes spent waiting in line.

(a) Verify that this is in fact a probability density function.

**Answer.** This function is clearly positive or zero and so there’s not much to do here other than compute the integral and show that it is equal to 1.

\[
\int_{-\infty}^{\infty} f(x) \, dx = \int_{0}^{\infty} 0.1e^{-\frac{x}{10}} \, dx = \lim_{c \to \infty} \int_{0}^{c} 0.1e^{-\frac{x}{10}} \, dx = \lim_{c \to \infty} (0.1)(10) \left(-e^{-\frac{c}{10}}\right)_{0}^{c} \\
= \lim_{c \to \infty} \left(-e^{-\frac{c}{10}}\right)_{0}^{c} = \lim_{c \to \infty} \left(e^{0} - e^{-\frac{c}{10}}\right) = 1 - 0 = 1.
\]

So it is a probability density function.

(b) Determine the probability that a person will wait in line for at least 6 minutes.

**Answer.** The probability that we are looking for here is \( P(X \geq 6) \).

\[
P(X \geq 6) = \int_{6}^{\infty} f(x) \, dx = \lim_{c \to \infty} \int_{6}^{c} 0.1e^{-\frac{x}{10}} \, dx = \lim_{c \to \infty} (0.1)(10) \left(-e^{-\frac{c}{10}}\right)_{6}^{c} \\
= \lim_{c \to \infty} \left(-e^{-\frac{c}{10}}\right)_{6}^{c} = \lim_{c \to \infty} \left(e^{-\frac{6}{10}} - e^{-\frac{c}{10}}\right) \\
= e^{-\frac{6}{10}} - 0 = e^{-\frac{3}{5}} \approx 0.5488116361.
\]

(c) What is the distribution function of \( X \)?

**Answer.** Since \( F(x) = 0 \) for \( x < 0 \), and

\[
F(x) = \int_{0}^{x} f(u) \, du = \int_{0}^{x} 0.1e^{-\frac{u}{10}} \, du = \left[-e^{-\frac{u}{10}}\right]_{0}^{x} = 1 - e^{-\frac{x}{10}}
\]

for \( x \geq 0 \), then we have

\[
F(x) = \begin{cases} 
0 & \text{if } x < 0, \\
1 - e^{-\frac{x}{10}} & \text{if } x \geq 0.
\end{cases}
\]
Question 5.  

Is the function 

\[ F(x, y) = \begin{cases} 
1 - e^{-x-y} & \text{if } x \geq 0, \ y \geq 0, \\
0 & \text{otherwise}, 
\end{cases} \]

the joint distribution function of some pair of continuous random variables?  

**Answer.** No, because \( F \) is twice differentiable with 

\[ f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} (1 - e^{-x-y}) \right) = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} e^{-x-y} \right) \]

\[ = -e^{-x-y} < 0 \quad \text{for } x \geq 0, \ y \geq 0. \]

So that \( F \) is not a joint distribution function, since it has negative values for \( x \geq 0, \ y \geq 0. \)

Question 6.  

The joint probability density function of \( X \) and \( Y \) is given by 

\[ f(x, y) = \begin{cases} 
\frac{6}{7} \left( x^2 + \frac{xy}{2} \right) & \text{if } 0 < x < 1, \ 0 < y < 2, \\
0 & \text{otherwise}. 
\end{cases} \]

(a) Compute the marginal density function of \( X \).  

**Answer.** 

\[ f_1(x) = \int_{y=-\infty}^{x=\infty} f(x, v) \, dv = \int_{v=0}^{2} \frac{6}{7} \left( x^2 + \frac{xv}{2} \right) \, dv = \frac{6}{7} \left( x^2 v + \frac{xv^2}{4} \right) \bigg|_{v=0}^{2} \]

\[ = \frac{6}{7} \left( 2x^2 + \frac{4x}{4} \right) = \frac{6x}{7}(2x + 1). \]

(b) Find \( P(X > Y) \).  

**Answer.** 

\[
\begin{align*}
P(X > Y) &= \int_{x=0}^{1} \int_{y=0}^{x} f(x, y) \, dy \, dx = \int_{x=0}^{1} \int_{y=0}^{x} \frac{6}{7} \left( x^2 + \frac{xy}{2} \right) \, dy \, dx \\
&= \frac{6}{7} \int_{x=0}^{1} \left( x^2 y + \frac{x^2 y^2}{4} \right) \bigg|_{y=0}^{x} \, dx = \frac{6}{7} \int_{x=0}^{1} \left( x^3 + \frac{x^3}{4} \right) \, dx \\
&= \frac{6}{7} \int_{x=0}^{1} \frac{5x^3}{4} \, dx = \frac{15x^4}{14\ 4} \bigg|_{x=0}^{1} = \frac{15}{56} \approx 0.2678571429.
\end{align*}
\]
A box contains two red pens, one green pen, and one blue pen, and you choose two pens without replacement. Let $X$ be the number of red pens that you choose and $Y$ the number of green pens. Find the joint probability distribution of $X$ and $Y$.

**Answer.** There are $\binom{4}{2} = \frac{4!}{2!2!} = 6$ different ways to make this selection, and here are the probabilities:

- $f(0, 0) = P(X = 0, Y = 0) = 0$ (since there is only one blue pen)
- $f(0, 1) = P(X = 0, Y = 1) = \frac{\binom{2}{1}\binom{2}{1}}{6} = \frac{1}{6}$, $f(1, 0) = P(X = 1, Y = 0) = \frac{\binom{2}{1}\binom{2}{1}}{6} = \frac{2}{6}$
- $f(1, 1) = P(X = 1, Y = 1) = \frac{\binom{2}{1}\binom{2}{1}}{6} = \frac{2}{6}$, $f(2, 0) = P(X = 2, Y = 0) = \frac{\binom{2}{1}\binom{2}{1}}{6} = \frac{4}{6}$
- $f(2, 1) = P(X = 2, Y = 1) = 0$ (since you choose only two pens).

So, the joint probability distribution of $X$ and $Y$ is

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