Directions — You have 105 minutes to complete the exam. Please do not leave the examination room in the first half of the exam. There are seven questions, of varying credit (100 points total). You must show your working to get full marks for a question, an answer alone will not earn credit. Indicate clearly your final answer to each question. You are allowed to use a calculator. During the exam, please turn off your cell phone(s). You cannot use any book or your notes. You have one double-sided page for “cheat-sheet” notes at the end of the exam papers. The answer key to this exam will be posted on Department of Mathematics and Computer Science board after the exam. Good luck!

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Question 1. 7 + 13 points

Let $X$ and $Y$ have joint density function

$$f(x, y) = \begin{cases} k, & \text{if } 0 \leq x, \text{ and } 0 \leq y \leq 1 - x, \\ 0, & \text{otherwise}, \end{cases}$$

where $k$ is a real constant.

(a) Find the value of $k$.

Answer.

![Diagram of the region of integration for $x + y = 1$.]

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = \int_{y=0}^{1} \int_{x=0}^{1-y} k \, dx \, dy = k \int_{y=0}^{1} x \bigg|_{x=0}^{1-y} \, dy$$

$$= k \int_{y=0}^{1} (1 - y) \, dy = k \left( y - \frac{y^2}{2} \right)_{0}^{1} = k \left( 1 - \frac{1}{2} \right) = k \frac{1}{2} \Rightarrow$$

$$k = 2.$$

(b) Are $X$ and $Y$ independent? (Justify your answer!)

Answer. If they $X$ and $Y$ are independent, then $f(x, y) = f_1(x) f_2(y)$.

$$f_1(x) = \int_{v=-\infty}^{\infty} f(x, v) \, dv = \int_{v=0}^{1-x} 2 \, dv = 2v \bigg|_{v=0}^{1-x} = 2(1 - x)$$

$$f_2(y) = \int_{u=-\infty}^{\infty} f(u, y) \, du = \int_{u=0}^{1-y} 2 \, du = 2u \bigg|_{u=0}^{1-y} = 2(1 - y)$$

$$f_1(x) = \begin{cases} 2(1 - x), & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases} \quad f_2(y) = \begin{cases} 2(1 - y), & 0 \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Since

$$f(x, y) = 2 \neq 4(1 - x)(1 - y) = f_1(x) f_2(y)$$

they are not independent. In particular, for $x = 0.2$ and $y = 0.3$ we see that

$$f(0.2, 0.3) = 2 \neq 2.24 = f_1(0.2) f_2(0.3) = 4(1 - 0.2)(1 - 0.3).$$
Question 2. 15 points

Suppose that $X$ and $Y$ have joint probability function as shown in the table below:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>-2</th>
<th>0</th>
<th>1</th>
<th>3.1</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>0.1</td>
<td>0.2</td>
<td>0.0</td>
<td>0.0</td>
<td>0.3</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>0.1</td>
<td>0.0</td>
<td>0.4</td>
<td>0.0</td>
<td>0.5</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td>0.0</td>
<td>0.1</td>
<td>0.0</td>
<td>0.1</td>
<td>0.2</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>0.2</td>
<td>0.3</td>
<td>0.4</td>
<td>0.1</td>
<td>1</td>
</tr>
</tbody>
</table>

Compute $\text{Cov}(X,Y)$.

**Answer.** Since $\text{Cov}(X,Y) = E[XY] - E[X]E[Y] = E[XY] - \mu_X\mu_Y$, we have

$E[X] = \mu_X = \sum_x \sum_y x f(x,y) = \sum_x x \left( \sum_y f(x,y) \right) = 1 \cdot 0.3 + 4 \cdot 0.5 + 9 \cdot 0.2 = 4.1$,

$E[Y] = \mu_Y = \sum_y \sum_x y f(x,y) = \sum_y y \left( \sum_x f(x,y) \right) = -2 \cdot 0.2 + 0 \cdot 0.3 + 1 \cdot 0.4 + 3.1 \cdot 0.1 = 0.31$,

$E[XY] = \sum_x \sum_y xy f(x,y) = -2 \cdot 1 \cdot 0.1 + -2 \cdot 4 \cdot 0.1 + 1 \cdot 4 \cdot 0.4 + 3.1 \cdot 9 \cdot 0.1 = 3.39$.

Therefore, we obtain that

$\text{Cov}(X,Y) = 3.39 - 0.31 \cdot 4.1 = 2.19$.

Question 3. 10 points

Suppose that $X$ and $Y$ are independent, with $\text{Var}[X] = 6$, $E[X^2Y^2] = 150$, $E[Y] = 2$ and $E[X] = 2$. Compute $\text{Var}(Y)$.

**Answer.** We know that


Since $X$ and $Y$ are independent, we have $E[XY] = E[X]E[Y]$ thus $E[X^2Y^2] = E[X^2]E[Y^2]$. Therefore,


Also, since $\text{Var}[Y] = E[Y^2] - (E[Y])^2$, then we obtain that

A continuous random variable $X$ has probability density function

$$f(x) = \begin{cases} 
3x^2, & 0 \leq x \leq 1 \\
0, & \text{otherwise}. 
\end{cases}$$

Find the value of $a$ such that $P(X \leq a) = P(X > a)$.

**Answer.** Note that $P(X > a) = 1 - P(X \leq a)$. Then,

$$P(X \leq a) = P(X > a) = 1 - P(X \leq a) \Rightarrow 2P(X \leq a) = 1 \Rightarrow P(X \leq a) = \frac{1}{2}.$$ 

Therefore, we would like to find the value of $a$ which satisfy the equality $P(X \leq a) = \frac{1}{2}$.

$$\frac{1}{2} = P(X \leq a) = \int_{-\infty}^{a} f(x) dx = \int_{0}^{a} 3x^2 dx = x^3 \bigg|_{0}^{a} = a^3 \Rightarrow a = \frac{1}{\sqrt{2}}.$$

**Question 5.**

A random variable $X$ has the following probability function

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>0</td>
<td>$k$</td>
<td>$2k$</td>
<td>$2k$</td>
<td>$3k$</td>
<td>$k^2$</td>
<td>$2k^2$</td>
<td>$7k^2 + k$</td>
</tr>
</tbody>
</table>

Find

(a) The value of $k$.

**Answer.**

$$1 = \sum_{x=0}^{7} f(x) = 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 10k^2 + 9k \Rightarrow$$

$$10k^2 + 9k - 1 = 0 \Rightarrow k = -1 \text{ and } k = 1/10.$$ 

Since $f(x) \geq 0$ for all $x = 0, 1, \cdots, 7$ then $k$ cannot be $-1$. So, we obtain that $k = 1/10$.

(b) $P(1.5 < X < 4.5 | X > 2)$.

**Answer.**

$$P(1.5 < X < 4.5 | X > 2) = \frac{P(X = 3) + P(X = 4)}{1 - P(X \leq 2)} = \frac{P(X = 3) + P(X = 4)}{1 - (P(X = 0) + P(X = 1) + P(X = 2))} = \frac{2k + 3k}{1 - (0 + k + 2k)} = \frac{5k}{1 - 3k} = \frac{5 \cdot \frac{1}{10}}{1 - 3 \cdot \frac{1}{10}} = \frac{5}{7}.$$
Question 6. 10 points

Given the joint density function

\[
f(x, y) = \begin{cases} \frac{x(1+3y^2)}{4}, & 0 < x < 2, 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}
\]

Find \( f(x|y) \).

**Answer.** Since \( f(x|y) = \frac{f(x,y)}{f_2(y)} \), then if \( 0 < y < 1 \)

\[
f_2(y) = \int_{u=-\infty}^{\infty} f(u, y) du = \int_{u=0}^{2} \frac{u(1+3y^2)}{4} du = \frac{(1+3y^2)}{8} u^2|_{u=0}^{u=2} = \frac{(1+3y^2)}{2},
\]

elsewhere \( f_2(y) = 0 \). Thus, if \( 0 < y < 1 \)

\[
f(x|y) = \frac{x(1+3y^2)}{(1+3y^2)} = \frac{x}{2} \quad \text{for} \quad 0 < x < 2,
\]

elsewhere \( f(x|y) = 0 \). For other values of \( y \) since \( f_2(y) = 0 \), then \( f(x|y) \) is undefined.

Question 7. 10 + 10 points

(a) Find the moment generation function of the random variable whose probability density function is given by

\[
f(x) = \begin{cases} 3e^{-3x}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}
\]

**Answer.**

\[
M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x) dx = \int_{0}^{\infty} e^{tx} 3e^{-3x} dx
\]

\[
= 3 \int_{0}^{\infty} e^{x(t-3)} dx = \left. \frac{3e^{x(t-3)}}{t-3} \right|_{0}^{\infty} = \frac{3}{3-t} \text{ assuming } t < 3.
\]

(b) Use the moment generating function to find the first 3 moments about the origin.

**Answer.** Since

\[
M_X(t) = \frac{3}{3-t} = \frac{1}{1 - \frac{t}{3}} = 1 + \frac{t}{3} + \frac{t^2}{3^2} + \frac{t^3}{3^3} + \cdots \text{ for } |t/3| < 1 \text{ or } |t| < 3
\]

and

\[
M_X(t) = 1 + \mu'_1 t + \mu'_2 \frac{t^2}{2!} + \mu'_3 \frac{t^3}{3!} + \cdots
\]

then

\[
\mu'_1 = \frac{1}{3}, \quad \mu'_2 = \frac{2}{9}, \quad \mu'_3 = \frac{2}{9}.
\]