Directions – You have 105 minutes to complete the exam. Please do not leave the examination room in the first half of the exam. There are seven questions, of varying credit (100 points total). You must show your working to get full marks for a question, an answer alone will not earn credit. Indicate clearly your final answer to each question. You are allowed to use a calculator. During the exam, please turn off your cell phone(s). You cannot use any book or your notes. The answer key to this exam will be posted on Department of Mathematics and Computer Science board after the exam. Good luck!

Emel Yavuz Duman, Ph.D.

Question 1.

Two symmetric dice have both had two of their sides painted red, two painted black, one painted yellow, and the other painted white. When this pair of dice are flipped, what is the probability that both land on the same color?

Solution.

\[
P(\text{both land on the same color}) = \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6} + \frac{2}{6} \times \frac{2}{6} + \frac{2}{6} \times \frac{2}{6} = \frac{1 + 1 + 4 + 4}{36} = \frac{10}{36} = \frac{5}{18}
\]

Question 2.

Let \( X \) be a random variable with mean \( \mu = -4 \). By using the Chebyshev’s inequality, find the variance of \( X \), if it is given that \( P(-10 < X < 2) \geq 0.5 \).

Solution. We want to apply the Chebyshev’s inequality that says that

\[
P(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}.
\]

We have \( P(-10 < X < 2) \geq 0.5 \). We start by adding \(-\mu\) to every part of the inequality to get

\[
P(-10 - (-4) < X - \mu < 2 - (-4)) \geq 0.5 \Rightarrow P(-6 < X - \mu < 6) \geq 0.5
\]

We see that we can rewrite this as \( P(|X - \mu| < 6) \geq 0.5 \). Since

\[
P(|X - \mu| < 6) = 1 - P(|X - \mu| \geq 6)
\]

then

\[
1 - P(|X - \mu| \geq 6) \geq 0.5 \Rightarrow P(|X - \mu| \geq 6) \leq 0.5.
\]

If we take \( \varepsilon = 6 \) in the Chebyshev’s inequality, we obtain that

\[
P(|X - \mu| \geq 6) \leq 0.5 = \frac{\sigma^2}{6^2} \Rightarrow \sigma^2 = Var(X) = 36 \times 0.5 = 18.
\]
During the noon lunch hour, 47 customers will walk through the door of the central post office. Assume that each person arrives at a random time, independent of the other customers. What is the probability that more than one person walks through the door during the first minute?

**Hint:** Use the method of Poisson approximation to the binomial distribution.

**Solution.** To see why this is binomial, think of each of the 47 persons choosing at random a minute during which to arrive. The probability that they choose the first minute is then 1/60. Thus, we have 47 repetitions of a trial, where the probability of success is 1/60. The mean number of successes is

$$\mu = 47 \times \frac{1}{60} \approx 0.78.$$ 

To find the probability, we say

$$P(\text{more than 1 arrival in 1st minute}) = 1 - P(0 \text{ or 1 arrival in 1st minute}).$$

$$P(0 \text{ or 1 arrival in 1st minute}) = \frac{0.78^0}{0!}e^{-0.78} + \frac{0.78^1}{1!}e^{-0.78} \approx 0.8159627001.$$

$$P(\text{more than 1 arrival in 1st minute}) = 1 - 0.8159627001 \approx 0.1840372999.$$

---

**Question 4.**

Tomorrow morning’s Iberia flight to Madrid can seat 370 passengers. From past experience, Iberia knows that the probability is 0.90 that a given ticket-holder will show up for the flight. They have sold 400 tickets, deliberately overbooking the flight. How confident can Iberia be that no passenger will need to be “bumped” (denied boarding)? **Hint:** Use the method of Normal approximation to the binomial distribution.

**Solution.** We will assume that the number \(X\) of passengers showing up for the flight has a binomial distribution with mean \(\mu = 400 \times 0.9 = 360\) and standard deviation \(\sigma = \sqrt{400 \times 0.9 \times 0.1} = 6\).

We want to find that \(P(X \leq 370)\). We approximate this by the probability that our normal random variable is less than 370.5. This is the probability that a standard normal is less than

$$Z = \frac{370.5 - 360}{6} = 1.75.$$

So the probability that nobody gets bumped is approximately

$$P(Z \leq 1.75) = 0.5 + 0.4599 = 0.9599 \ (\text{Almost 96\%}).$$
Let the joint density of two random variables $X$ and $Y$ be given by
\[
f(x, y) = \begin{cases} \frac{1}{4}(2x + y), & 0 \leq x \leq 1, \ 0 \leq y \leq 2, \\ 0, & \text{otherwise.} \end{cases}
\]

Find the conditional mean (or expectation) of $X$ given $Y = 0.5$.

**Solution.** Since the conditional mean of $X$ given $Y$ is
\[
\mu^*_1 = E[X|Y = y] = \int_{-\infty}^{\infty} xf(x|y)dx
\]
we need to calculate $f(x|y) = \frac{f(x,y)}{f_2(y)}$, therefore $f_2(y) = \int_{-\infty}^{\infty} f(u,y)du$. Since
\[
f_2(y) = \begin{cases} \int_{-\infty}^{\infty} f(u,y)du = \int_{0}^{1} \frac{1}{4}(2u + y)du = \frac{1}{4}(u^2 + uy)_0^1 = \frac{1}{4}(1 + y), & 0 \leq y \leq 2, \\ 0, & \text{otherwise,} \end{cases}
\]
we have that, for $0 \leq y \leq 2$
\[
f(x|y) = \begin{cases} \frac{f(x,y)}{f_2(y)} = \frac{\frac{1}{4}(2x+y)}{\frac{1}{4}(1+y)} = \frac{2x+y}{1+y}, & 0 \leq x \leq 1, \\ 0, & \text{other } x. \end{cases}
\]
The conditional density function is not defined when $f_2(y) = 0$. So,
\[
f(x|0.5) = \frac{2x + 0.5}{1 + 0.5} = \frac{4x + 1}{3}.
\]
Therefore, we obtain that the conditional mean of $X$ given $Y = 0.5$ as
\[
\mu^*_1 = E[X|Y = 0.5] = \int_{-\infty}^{\infty} xf(x|0.5)dx = \int_{0}^{1} x \left( \frac{4x + 1}{3} \right) dx = \frac{2}{3} \left( \frac{4}{3} x^3 + \frac{1}{2} x^2 \right)_0^1 = \frac{11}{18}.
\]

**Question 6.**

Let $X$ be a random variable with mean $\mu$ and $c$ be any constant. Show that
\[
E[(X - c)^2] = Var(X) + (c - \mu)^2.
\]

**Proof.**
\[
E[(X - c)^2] = E[(X - c + \mu - \mu)^2] = E[(X - \mu) - (c - \mu)] \nonumber^2
\]
\[
= E[(X - \mu)^2 + (c - \mu)^2 - 2(X - \mu)(c - \mu)]
\]
\[
= \underbrace{E[(X - \mu)^2]}_{Var(X)} + (c - \mu)^2 - 2(c - \mu) \underbrace{E[(X - \mu)]}_{0}
\]
\[
= Var(X) + (c - \mu)^2.
\]
Consider the following random experiment: an urn contains 4 balls numbered from 1 to 4. Without looking, pick two balls from the urn (i.e., without replacing the first in the urn before choosing the second). Call the value of the first one \( X \) and the value of the second one \( Y \). Find \( \text{Cov}(X,Y) \) and \( \text{Var}(X+Y) \).

**Solution.** \( \text{Cov}(X,Y) = E(XY) - E(X)E(Y) \). Let us compute these quantities. \( X \) takes one of the 4 values 1, 2, 3, 4 with equal probability \( 1/4 \), so

\[
E(X) = \frac{1}{4}(1 + 2 + 3 + 4) = \frac{5}{2}
\]

and the same applies to \( Y \), i.e., \( E(Y) = 5/2 \). On the other hand, the couple \((X,Y)\) takes with equal probability \( 1/6 \) one of the following 6 values:

\[(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\]

Correspondingly, the product \( XY \) takes the following values:

\[2, 3, 4, 6, 8, 12,\]

so

\[
E(XY) = \frac{1}{6}(2 + 3 + 4 + 6 + 8 + 12) = \frac{35}{6}.
\]

Finally,

\[
\text{Cov}(X,Y) = \frac{35}{6} - \frac{5}{2} \times \frac{5}{2} = -\frac{5}{12}.
\]

Next, in order to find \( \text{Var}(X+Y) \) we use the fact that

\[
\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) - 2\text{Cov}(X,Y).
\]

Since

\[
\text{Var}(X) = \text{Var}(Y) = \frac{1}{4} \left( \left( 1 - \frac{5}{2} \right)^2 + \left( 2 - \frac{5}{2} \right)^2 + \left( 3 - \frac{5}{2} \right)^2 + \left( 4 - \frac{5}{2} \right)^2 \right) = \frac{5}{4}
\]

then

\[
\text{Var}(X+Y) = \frac{5}{4} + \frac{5}{4} - \left( 2 \times \frac{35}{6} \right) = \frac{5}{3}.
\]