FIRST MIDTERM

Fall 2012-2013

Number:
Name:
Department:

Directions
- You have 90 minutes to complete the exam. Please do not leave the examination room in the first 30 minutes of the exam. There are six questions, of varying credit (100 points total). Indicate clearly your final answer to each question. You are allowed to use a calculator. During the exam, please turn off your cell phone(s). You cannot use the book or your notes. You have one page for “cheat-sheet” notes at the end of the exam papers. The answer key to this exam will be posted on Department of Mathematics and Computer Science board after the exam.

Good luck!

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TOTAL
Question 1.  

10 + 10 + 10 points

(a) I roll a pair of (fair) dice 3 times. What is the probability that, on at least one of the three tries, I roll a 5?

(b) I pick three cards from a deck, without replacement. (Recall that in a deck of 52 cards, there are 13 cards of each suit.) In how many different ways can it happen that two are of the same suit and one is of a different suit, if order does’t matters?

(c) If eight persons are having dinner together, in how many different ways can three order chicken, four order steak, and one order lobster?

Answer.

(a) The probability that I fail to roll a 5, on any particular try, is \( \frac{5}{6} \cdot \frac{5}{6} = \left( \frac{5}{6} \right)^2 \). Hence the probability that I fail on all three tries is \( \left( \frac{5}{6} \right)^6 \) and the probability that I succeed on at least one try is \( 1 - \left( \frac{5}{6} \right)^6 \approx 0.6651020233 \).

(b) One can choose any of the 4 suits to be the long suit. Then one can choose two cards in the long suit in \( \binom{13}{2} = 78 \) ways, and one card not in the long suit in \( 52 - 13 = 39 \) ways. Hence the total number of such hands, if order does not matter, is \( \binom{13}{2} \cdot \binom{39}{1} \cdot 4 = 12,168 \).

(c) There are 8 dishes and three of them are chicken, four of them are steak, and one is lobster. So the number of different permutations is \( \frac{8!}{3! \cdot 4! \cdot 1!} = 280 \).

Question 2.  

10 points

Let \( A \) and \( B \) and two events such that \( P(A|B) < P(A) \), prove that \( P(B|A) < P(B) \).

Answer. If \( P(A|B) = \frac{P(A \cap B)}{P(B)} < P(A) \), then \( P(A \cap B) < P(A)P(B) \). So,

\[
P(B|A) = \frac{P(A \cap B)}{P(A)} \Rightarrow P(A|B)P(A) = P(A \cap B) < P(A)P(B)
\]

thus we find that

\[
P(B|A) < P(B).
\]

Question 3.  

10 points

The probability density function of the random variable \( X \) is given by

\[
f(x) = \begin{cases} 
x, & 0 < x < 1, \\
c - x, & 1 \leq x < 2, \\
0, & \text{elsewhere.}
\end{cases}
\]

Find the value of \( c \).

Answer.

\[
1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx
\]

\[
= \int_{0}^{1} xd\,x + \int_{1}^{2} (c - x)\,dx = \frac{x^2}{2} \bigg|_{0}^{1} + \left( cx - \frac{x^2}{2} \right) \bigg|_{1}^{2} = \frac{1}{2} + (2c - 2) - \left( c - \frac{1}{2} \right) \Rightarrow
\]

\[
c = 2.
\]
Question 4. 5 + 10 points

In a certain community, 8 percent of all adults over 50 have diabetes. If a health service in this community correctly diagnoses 95 percent of all persons with diabetes as having the disease and incorrectly diagnoses 2 percent of all persons without diabetes as having the disease, find the probabilities that,

(a) the community health service will diagnose an adult over 50 as having diabetes.
(b) a person over 50 diagnosed by the health service as having diabetes actually has the disease.

Answer. Let us first identify the events associated with this problem. \( A \) is the event of an adult over 50 diagnosed by the health service. \( B_1 \) is the event that an adult over 50 actually has the diabetes. So \( P(B_1) = 0.08 \). \( B_2 \) is the event that an adult over 50 does not have the diabetes. So, \( P(B_2) = 1 - P(B_1) = 1 - 0.08 = 0.92 \). From the available information, \( P(A|B_1) = 0.95 \) and \( P(A|B_2) = 0.02 \).

(a) \( P(B_1)P(A|B_1) + P(B_2)P(A|B_2) = (0.08)(0.95) + (0.92)(0.02) = 0.0944 \).

(b) \( P(B_1|A) = \frac{P(B_1)P(A|B_1)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2)} = \frac{(0.08)(0.95)}{(0.08)(0.95) + (0.92)(0.02)} = 0.8051 \).

Question 5. 10 points

An urn contains 10 balls: 4 red and 6 blue. A second urn contains 16 red balls and an unknown number of blue balls. A single ball is drawn from each urn. The probability that both balls are the same color is 0.44. Calculate the number of blue balls in the second urn.

Answer. The event “both balls have the same color”, which we know has probability 0.44, splits into two disjoint events, “ball 1 red and ball 2 red” and “ball 1 blue and ball 2 blue”. Since the drawings from the two urns are obviously independent, it follows that

\[ 0.44 = P(\text{ball 1 red}) \cdot P(\text{ball 2 red}) + P(\text{ball 1 blue}) \cdot P(\text{ball 2 blue}). \]

Since Urn 1 has 4 red balls and 6 blue balls, the probability for drawing a red ball from this urn is 4/10, and the probability for drawing a blue ball is 6/10. The corresponding probabilities for Urn 2 are \( 16/(16 + x) \), and \( x/(16 + x) \), where \( x \) denotes the number of blue balls in Urn 2. Thus we get

\[ 0.44 = \frac{4}{10} \cdot \frac{16}{16 + x} + \frac{6}{10} \cdot \frac{x}{16 + x}. \]

Solving this equation for \( x \) gives

\[ x = \frac{(0.44 - 0.4) \cdot 16}{0.6 - 0.44} = 4. \]
Question 6.  

Roll two fair, four-sided dice. Call the resulting two numbers \( m \) and \( n \). Let

\[
x = \max\{m, n\} \quad \text{and} \quad y = \min\{m, n\}.
\]

(a) Construct a table showing the values the joint probability distribution of \( X \) and \( Y \).

(b) Compute the marginal probability of \( X \).

(c) Compute the conditional probability of \( Y \) given \( X = 2 \).

Solution.

(a) Since there are 16 pairs with the probability \( 1/16 \)

\[
\begin{array}{c|cccc}
\text{Roll 1} & 1 & 2 & 3 & 4 \\
\hline
1 & (1,1) & (2,1) & (3,1) & (4,1) \\
2 & (2,1) & (2,2) & (3,2) & (4,2) \\
3 & (3,1) & (3,2) & (3,3) & (4,3) \\
4 & (4,1) & (4,2) & (4,3) & (4,4) \\
\end{array}
\]

then, the joint probability distribution of \( X \) and \( Y \) is

\[
\begin{array}{c|cccc}
\hline
\text{X} & \text{Y} & 1 & 2 & 3 & 4 \\
\hline
1 & 1/16 & 0 & 0 & 0 & 1/16 \\
2 & 2/16 & 1/16 & 0 & 0 & 3/16 \\
3 & 2/16 & 2/16 & 1/16 & 0 & 5/16 \\
4 & 2/16 & 2/16 & 1/16 & 1/16 & 7/16 \\
\hline
h(y) & 7/16 & 5/16 & 3/16 & 1/16 & 1 \\
\end{array}
\]

(b) The marginal probability of \( X \) is

\[
g(x) = \begin{cases} 
1/16, & x = 1, \\
3/16, & x = 2, \\
5/16, & x = 3, \\
7/16, & x = 4.
\end{cases}
\]

(c) The conditional probability of \( Y \) given \( X = 2 \) is

\[
P(Y|X = 2) = w(y|2) = f(2, y) = \frac{f(2, y)}{g(2)} = \frac{f(2, y)}{3/16} = \frac{16}{3} f(2, y),
\]

\[
w(y|2) = \begin{cases} 
\frac{16}{3} f(2, 1) = \frac{16}{3} \cdot \frac{2}{16} = \frac{2}{3}, & y = 1, \\
\frac{16}{3} f(2, 2) = \frac{16}{3} \cdot \frac{1}{16} = \frac{1}{3}, & y = 2, \\
\frac{16}{3} f(2, 3) = \frac{16}{3} \cdot 0 = 0, & y = 3, \\
\frac{16}{3} f(2, 4) = \frac{16}{3} \cdot 0 = 0, & y = 4.
\end{cases}
\]