SECOND MIDTERM

Fall 2012-2013

SOLUTIONS

Directions — You have 90 minutes to complete the exam. Please do not leave the examination room in the first 30 minutes of the exam. There are six questions, of varying credit (100 points total). Indicate clearly your final answer to each question. You are allowed to use a calculator. During the exam, please turn off your cell phone(s). You cannot use the book or your notes. You have one page for "cheat-sheet" notes at the end of the exam papers. The answer key to this exam will be posted on Department of Mathematics and Computer Science board after the exam.

Good luck! 

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Question 1.  

The joint probability distribution of $X$ and $Y$ is given by 

$$f(x, y) = \frac{x + y}{21}, \quad x = 1, 2, 3, \text{ and } y = 1, 2.$$ 

Find the conditional variance of $X$ given $Y = 1$. 

**Answer.** 

<table>
<thead>
<tr>
<th>$X$</th>
<th>$1$</th>
<th>$2$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h(y)$</td>
<td>$9/21$</td>
<td>$12/21$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

$$f(x|1) = \frac{f(x,1)}{h(1)} = \frac{f(x,1)}{\frac{21}{21}} = \frac{21}{9} f(x, 1) \Rightarrow f(1|1) = \frac{2}{9}, \quad f(2|1) = \frac{3}{9}, \quad f(3|1) = \frac{4}{9}.$$ 

$$E[X|1] = \mu_{X|1} = \sum_{x=1}^{3} x f(x|1) = 1 \cdot f(1|1) + 2 \cdot f(2|1) + 3 \cdot f(3|1) = 1 \cdot \frac{2}{9} + 2 \cdot \frac{3}{9} + 3 \cdot \frac{4}{9} = \frac{20}{9}.$$ 

$$E[X^2|1] = \sum_{x=1}^{3} x^2 f(x|1) = 1^2 \cdot f(1|1) + 2^2 \cdot f(2|1) + 3^2 \cdot f(3|1) = 1^2 \cdot \frac{2}{9} + 2^2 \cdot \frac{3}{9} + 3^2 \cdot \frac{4}{9} = \frac{50}{9}.$$ 

$$\sigma_{X|1} = E[X^2|1] - (E[X|1])^2 = \frac{50}{9} - \left(\frac{20}{9}\right)^2 = \frac{50}{81}.$$ 

Question 2.  

If $X$ is the number of heads and $Y$ is the number of heads minus the number of tails obtained in three flips of a balanced coin find the covariance of $X$ and $Y$. 

**Answer.** There are 8 possible outcomes: 

$$\begin{align*} 
HHH: & \quad X = 3, Y = 3; \\
HHT: & \quad X = 2, Y = 1; \\
HTH: & \quad X = 2, Y = 1; \\
HHT: & \quad X = 1, Y = -1; \\
HTT: & \quad X = 1, Y = -1; \\
TTT: & \quad X = 0, Y = -3; \\
TTH: & \quad X = 1, Y = -1; \\
TTH: & \quad X = 2, Y = +1.
\end{align*}$$

The the joint probability distribution is 

<table>
<thead>
<tr>
<th>$X$</th>
<th>$-3$</th>
<th>$-1$</th>
<th>$1$</th>
<th>$3$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h(y)$</td>
<td>$0$</td>
<td>$1/8$</td>
<td>$3/8$</td>
<td>$3/8$</td>
<td>$1/8$</td>
</tr>
</tbody>
</table>

Since 

$$E(XY) = \mu'_{1,1} = \sum_{x} \sum_{y} xy f(x, y) = 0(-3)\frac{1}{8} + 1(-1)\frac{3}{8} + 2(1)\frac{3}{8} + 3(3)\frac{1}{8} = \frac{12}{8},$$

$$E(X) = \mu_{x} = \sum_{x} x g(x) = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{12}{8},$$

$$E(Y) = \mu_{y} = \sum_{y} y h(y) = (-3) \cdot \frac{1}{8} + (-1) \cdot \frac{3}{8} + 1 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = 0,$$

thus 

$$\sigma_{XY} = E(XY) - E(X)E(Y) = \mu'_{1,1} - \mu_{x}\mu_{y} = \frac{12}{8} - \frac{12}{8} \cdot 0 = \frac{12}{8} = \frac{3}{2}.$$
**Question 3.**

The number of marriage licenses issued in a certain city during the month of June may be looked upon as a random variable with \( \mu = 124 \) and \( \sigma = 7.5 \). According to Chebyshev’s theorem, with what part probability can we assert that between 64 and 184 marriage licenses will be issued there during the month of June?

**Answer.** Since \( 64 < X < 184 \), then

\[
64 - 124 < X - 124 < 184 - 124 \Rightarrow -60 < X - 124 < 60 \Rightarrow |X - 124| < 60.
\]

Therefore,

\[
P(|X - 124| < 60) = 1 - P\left(\frac{|X - 124|}{\mu} \geq \frac{60}{\varepsilon}\right) \geq 1 - \frac{\sigma^2}{\varepsilon^2} = 1 - \frac{7.5^2}{60^2} = 1 - \frac{1}{64} = \frac{63}{64}.
\]

So, the probability of between 64 and 184 marriage licenses will be issued there during the month of June is at least \( \frac{63}{64} \).

**Question 4.**

The probability density function of a random variable \( X \) is given by

\[
f(x) = \begin{cases} 
1/3, & 0 < x \leq 1, \\
2/3, & 1 < x \leq 2, \\
0, & \text{otherwise}.
\end{cases}
\]

Find the corresponding distribution function of \( X \) and use it to calculate \( P(0.5 < X < 1.7) \).

**Answer.** Since

- If \( x < 0 \) then \( F(x) = \int_{-\infty}^{x} f(t)dt = 0 \)
- If \( 0 < x \leq 1 \) then \( F(x) = \int_{-\infty}^{x} f(t)dt = \int_{0}^{x} \frac{1}{3} dt = \frac{1}{3} x \)
- If \( 1 < x \leq 2 \) then \( F(x) = \int_{-\infty}^{x} f(t)dt = \int_{0}^{1} \frac{1}{3} dt + \int_{1}^{x} \frac{2}{3} dt = \frac{1}{3} + \frac{2}{3} (x - 1) = \frac{2}{3} x - \frac{1}{3} \)
- If \( x > 2 \) then \( F(x) = \int_{-\infty}^{x} f(t)dt = 1 \)

therefore

\[
F(x) = \begin{cases} 
0, & x < 0, \\
\frac{x}{3}, & 0 < x \leq 1, \\
\frac{2x-1}{3}, & 1 < x \leq 2, \\
1, & x > 2,
\end{cases}
\]

and

\[
P(0.5 < X < 1.7) = F(1.7) - F(0.5) = \frac{2(1.7) - 1}{3} - \frac{0.5}{3} = \frac{19}{30} = 0.63.
\]
Question 5.  
Let $X$ be the number on a fair six sided die roll.

(a) Find the moment generating function $M_X(t)$.

(b) Use the moment generating function of $X$ to determine $\mu'_1$ and $\mu'_2$.

Answer.

(a) $M_X(t) = E(e^{tX}) = \sum_{x=1}^{6} e^{tx} f(x) = \frac{1}{6}(e^{t} + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t})$.

(b) $\mu'_1 = \mu = M'_X(t)|_{t=0} = \frac{1}{6}(e^{t} + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t})|_{t=0}$

$= \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \frac{21}{6} = \frac{7}{2}$

$\mu'_2 = M''_X(t)|_{t=0} = \frac{1}{6}(e^{t} + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t})|_{t=0}$

$= \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) = \frac{91}{6}$.

Question 6.  
The distribution function of the continuous random variable $X$ is given by

$$F(x) = \begin{cases} 
0 & \text{for } x < 0, \\
x/10 & \text{for } 0 \leq x \leq 10 \\
1 & \text{for } x > 10.
\end{cases}$$

Find the standard deviation of $X$.

Answer. We first need to find the probability density of $X$:

$$f(x) = \frac{d}{dx} F(x) = \begin{cases} 
\frac{d}{dx} \frac{0}{0} = 0 & \text{for } x < 0, \\
\frac{d}{dx} \frac{x}{10} = \frac{1}{10} & \text{for } 0 < x < 10, \\
\frac{d}{dx} 1 = 0 & \text{for } x > 10.
\end{cases}$$

Since

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_{0}^{10} x^2 \cdot \frac{1}{10} dx = \frac{1}{10} \left[ \frac{x^3}{3} \right]_0^{10} = \frac{10^3}{10 \cdot 3} = \frac{100}{3},$$

and

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_{0}^{10} x \cdot \frac{1}{10} dx = \frac{1}{10} \left[ \frac{x^2}{2} \right]_0^{10} = \frac{10^2}{10 \cdot 2} = \frac{10}{2},$$

then

$$\sigma = \sqrt{\sigma^2} = \sqrt{E(X^2) - (E(X))^2} = \sqrt{\frac{100}{3} - \left( \frac{10}{2} \right)^2} = \frac{5\sqrt{3}}{3} \approx 2.8868.$$