Quantity Discounts

- EOQ models assume that unit cost / purchasing price is constant, no matter what quantity purchased.
- Sometimes, suppliers may offer discounted price (unit cost) for the larger quantity purchased (Quantity Discounts).
- EOQ model can be extended for involving Quantity Discounts.
Objective

Balance the tradeoff between savings by purchasing at a lower price and higher inventory holding cost due to larger quantity purchased.

Discount Schedules

- **All – units discount**: Applies the discounted price to all units purchased.

- **Incremental discount**: Applies the discounted price only to those units over the price break quantity.
All – units Discount

Applies the discounted price to all units purchased.

\[ C_j(Q) = c_j Q \quad \text{for } b_{j-1} \le Q < b_j \]

Incremental Discount

Applies the discounted price only to those units over the price break quantity.

\[ C_j(Q) = \sum_{k=1}^{j-1} c_k (b_k - b_{k-1}) + c_j (Q - b_{j-1}) \quad \text{for } b_{j-1} \le Q < b_j \]

Where,
\[ C_j(Q) \] = purchasing cost of lot size \( Q \) at price break interval \( j \).
\( b_j \) = upper limit of the \( j^{\text{th}} \) price break interval \([b_{j-1}, b_j]\).
\( c_j \) = cost of one unit in the \( j^{\text{th}} \) price break interval.
Example

For certain component used in your company, there are two suppliers (A and B). Both suppliers (A and B) offer a discount policy based on order quantity as given below:

*C(Q) for two price schedules:*

<table>
<thead>
<tr>
<th>Quantity (Q)</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 ≤ Q &lt; 500</td>
<td>0.6Q</td>
<td>0.6Q</td>
</tr>
<tr>
<td>500 ≤ Q &lt; 1000</td>
<td>0.58Q</td>
<td>0.6(500) + 0.58(Q − 500)</td>
</tr>
<tr>
<td>1000 ≤ Q &lt; ∞</td>
<td>0.56Q</td>
<td>0.6(500) + 0.58(500) + 0.56(Q − 1000)</td>
</tr>
</tbody>
</table>

\[ i = 20\% \text{ per unit per year} \quad A = \$20 /\text{order} \]
\[ D = 2000 \text{ units per year} \]
C(Q) for Incremental Discount Policy

EOQ for All-units discount policy

\[ TC_j(Q) = c_j D + H_j \frac{Q}{2} + \frac{AD}{Q} \]

\[ \Rightarrow EOQ_j = Q_j = \sqrt{\frac{2AD}{H_j}} \]

\( TC_j(Q) \) = total cost for price j
\( H_j = i(c_j) \) = inventory holding cost at price j
Procedure to find EOQ

1. Find EOQ for each price break interval, if EOQ\textsubscript{j} is in the interval of \([b_{j-1}, b_{j})\) take this EOQ\textsubscript{j} as the optimal lot size of the interval (Q\textsuperscript{*}\textsubscript{j}), if it is outside from the interval take the closest value in the interval as optimal lot size.

2. Compute TC\textsubscript{j}(Q) for all price break intervals.

3. Select the lot size Q\textsuperscript{*} with the minimum TC\textsubscript{j}(Q) as being EOQ over all prices.

EOQ for All-units discount policy

![Graph showing EOQ for different all-units discount policies with c\textsubscript{1} = 0.6, c\textsubscript{2} = 0.58, and c\textsubscript{3} = 0.56]
EOQ for All-units discount policy

\[ EOQ_j = Q_j = \sqrt{\frac{2AD}{H_j}} \]

**EOQ\_1 for H\_1 = 0.6(0.2) = 0.12**
\[ Q_1 = 817 \text{ units} > 500 \text{ then set } Q^{*}_1 = 499 \text{ units} \]

**EOQ\_2 for H\_2 = 0.58(0.2) = 0.116**
\[ 500 < Q_2 = 830 \text{ units} < 1000 \text{ then set } Q^{*}_2 = 830 \text{ units} \]

**EOQ\_3 for H\_3 = 0.56(0.2) = 0.112**
\[ Q_3 = 845 \text{ units} < 1000 \text{ then set } Q^{*}_3 = 1000 \text{ units} \]

---

EOQ for All-units discount policy

For \( Q^{*}_2 = 1000 \text{ units}, c_3 = 0.56, H_3 = 0.112 \)
\[ TC_3(1000) = 0.56(2000) + 0.112 \frac{1000}{2} + 20 \frac{2000}{1000} = $1216 \]

For \( Q^{*}_2 = 830 \text{ units}, c_2 = 0.58, H_2 = 0.116 \)
\[ TC_2(830) = 0.58(2000) + 0.116 \frac{830}{2} + 20 \frac{2000}{830} = $1256 \]
\[ TC_2(Q) > TC_3(Q) \text{ then} \]
EOQ over all prices \( Q^* = Q^{*}_3 = 1000 \text{ units} \)
and \( TC(Q^*) = $1216 \text{ at the unit price of } $0.56 \)
EOQ for Incremental Discount Policy

\[ TC_j(Q) = \left[ \frac{C_j(Q)}{Q} \right] D + i \left[ \frac{C_j(Q)}{Q} \right] \frac{Q}{2} + \frac{AD}{Q} \]

\[ \Rightarrow EOQ_j = Q_j = \sqrt{\frac{2D (A + C(b_{j-1}) - c_j b_{j-1})}{H_j}} \]

\[ \frac{C_j(Q)}{Q} \] = unit cost for lot size Q at price break interval j.

\[ C(b_{j-1}) \] = total purchasing cost at price break interval j.

Procedure to find EOQ

1. Find EOQ for each price break interval, if EOQ\(_j\) is in the interval of \([b_{j-1}, b_j]\) take this EOQ\(_j\) as the optimal lot size of the interval (Q\(_j^*\)), otherwise no need to consider this EOQ\(_j\) in finding EOQ over all prices.

2. Compute TC\(_j\)(Q) for EOQ\(_j\)'s in the interval.

3. Select the lot size Q\(_j^*\) with the minimum TC\(_j\)(Q) as being EOQ over all prices.
**EOQ for Incremental Discount Policy**

\[ \frac{C_1(Q)}{Q} = 0.6 \]
\[ \frac{C_2(Q)}{Q} = \frac{10}{Q} + 0.58 \]
\[ \frac{C_3(Q)}{Q} = \frac{30}{Q} + 0.56 \]

\[ Q_{EOQ} = 0 \leq Q < 500 \]
\[ Q_{EOQ} = 500 \leq Q < 1000 \]
\[ Q_{EOQ} = 1000 \leq Q < \infty \]

\[ EOQ_j = Q_j = \sqrt{\frac{2D(A + C(b_{j-1}) - c_j b_{j-1})}{H_j}} \]

\[ EOQ_1 = Q_1 = \sqrt{\frac{2(2000)(20 + 0 - 0)}{0.12}} = 817 \text{ units} > 500 \]

*eliminate!*
## EOQ for Incremental Discount Policy

<table>
<thead>
<tr>
<th>j</th>
<th>$Q$</th>
<th>$C(b_{j-1})$</th>
<th>$c_j b_{j-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$0 \leq Q &lt; 500$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$500 \leq Q &lt; 1000$</td>
<td>$(0.6)500 = 300$</td>
<td>290</td>
</tr>
<tr>
<td>3</td>
<td>$1000 \leq Q &lt; \infty$</td>
<td>300+ $(0.58)500 = 590$</td>
<td>560</td>
</tr>
</tbody>
</table>

$$EOQ_j = Q_j = \sqrt{\frac{2D(A + C(b_{j-1}) - c_j b_{j-1})}{H_j}}$$

$$EOQ_2 = Q_2 = \sqrt{\frac{2(2000)(20 + 300 - 290)}{0.116}} = 1017 \text{ units} > 1000 \quad \text{eliminate!}$$

## EOQ for Incremental Discount Policy

<table>
<thead>
<tr>
<th>j</th>
<th>$Q$</th>
<th>$C(b_{j-1})$</th>
<th>$c_j b_{j-1}$</th>
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</tr>
<tr>
<td>3</td>
<td>$1000 \leq Q &lt; \infty$</td>
<td>300+ $(0.58)500 = 590$</td>
<td>560</td>
</tr>
</tbody>
</table>

$$EOQ_3 = Q_3 = \sqrt{\frac{2(2000)(20 + 590 - 560)}{0.112}} = 1336 \text{ units} > 1000 \quad \text{consider!}$$
EOQ for Incremental Discount Policy

EOQ over all prices $= Q^* = 1336$ units

$$\frac{C_j(Q)}{Q} = \frac{30}{Q} + 0.56 = \frac{30}{1336} + 0.56 = \$0.582 \text{ / unit}$$

$$TC_j(Q) = \left[ \frac{C_j(Q)}{Q} \right] D + i \left[ \frac{C_j(Q)}{Q} \right] \frac{Q}{2} + AD$$

$$TC_3(1336) = 0.582(2000) + 0.2(0.582)\frac{1336}{2} + 20 \frac{2000}{1336}$$

$$TC_3(1336) = 1164 + 78 + 30 = \$1272$$

Multiple Item EOQ Models

The classical EOQ model is for a single item.* What happens when we have more than one item?

Answer:
• Simply calculate the EOQ of each item, if there is no interaction among items.

\[ i : 1, 2, \ldots, m \text{ (number of items)} \]

\[ EOQ_i = \sqrt{\frac{2A_i D_i}{H_i}} \]
Resource Constrained Multiple Item EOQ Models

• What if multiple items share common resources such as; Budget, Storage capacity, or both.

Then the \( EOQ_i = \sqrt{\frac{2A_i D_i}{H_i}} \) procedure is no longer adequate

because common resources are limited, and results may violate the resource constraints

Multiple Item-EOQ With One Constraint

Suppose that we have a budget with the investment capacity of C. Say – total investment in inventory shouldn’t exceed C dollars.

Resource constraint:

\[ \sum_{i=1}^{m} c_i Q_i \leq C \]

Where;
- \( i = 1,2,\ldots,m \) (number of items)
- \( Q_i \) = lot size of item i
- \( c_i \) = unit cost of item i
- \( C \) = maximum amount that will be invested
Multiple Item-EOQ With One Constraint

**Objective:**
To minimize the total annual inventory cost of all items,

\[ TC_i(Q) = c_iD + H_i \frac{Q_i}{2} + \frac{A_iD_i}{Q_i} \quad \text{total annual cost of item } i \]

\[ TC = \sum_{i=1}^{m} TC_i(Q) \quad \text{total annual cost of all items} \]

Subject to:

\[ \sum_{i=1}^{m} c_iQ_i \leq C \]

Lagrange Multiplier (\( \lambda \)) can be used to consider constraint in objective function (TC):

**Lagrangian Function:**

\[ TC(Q, \lambda) = \sum_{i=1}^{m} TC_i(Q) + \lambda \left[ \sum_{i=1}^{m} c_iQ_i - C \right] \]

The Lagrange multiplier acts as a penalty to reduce each \( Q^*_i \) to minimize cost while enforcing the constraint.
Solution Procedure

1. Solve the unconstrained problem:

Find:  \( EOQ_i = \sqrt{\frac{2A_i D_i}{H_i}} \) \quad \text{for } i = 1, 2, \ldots, m

If the constraint is satisfied this solution is the optimal one.

2. If this is not the case, set TC(Q, \( \lambda \)).

\[
TC(Q, \lambda) = \sum_{i=1}^{m} TC_i(Q) + \lambda \left[ \sum_{i=1}^{m} c_i Q_i - C \right]
\]

3. Obtain \( Q_i^* \) and \( \lambda^* \) by solving (m+1) equations given by:

\[
\frac{dTC(Q, \lambda)}{dQ_i} = 0 \quad \Rightarrow \quad -\frac{A_i D_i}{Q_i^2} + \frac{H_i}{2} + \lambda c_i = 0 \quad (1)
\]

\[
\frac{dTC(Q, \lambda)}{d\lambda} = 0 \quad \Rightarrow \quad \sum_{i=1}^{m} c_i Q_i = C \quad (2)
\]

4. Solve (1) for \( Q_i^* \)

\[
Q_i^* = \sqrt{\frac{2A_i D_i}{H_i + 2\lambda c_i}} \quad \text{for } i = 1, 2, \ldots, m
\]
Solution Procedure

5. Substitute $Q^*_i$ into (2):

$$Q^*_i = \frac{2A_iD_i}{H_i + 2\lambda c_i} \quad \text{for } i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} c_i Q_i = C$$

6. Solve for $\lambda^*$ and then determine $Q^*_i$ values

Example A

<table>
<thead>
<tr>
<th>Floppy Drive 1</th>
<th>Floppy Drive 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>A = $50</td>
<td>$50</td>
</tr>
<tr>
<td>i = 20%</td>
<td>20%</td>
</tr>
<tr>
<td>c = $50</td>
<td>$80</td>
</tr>
<tr>
<td>H = $10</td>
<td>$16</td>
</tr>
<tr>
<td>D = 250</td>
<td>484</td>
</tr>
</tbody>
</table>

Investment budget is limited to $5000.
Example A Continued

**Step 1:** Solve the unconstrained problem

\[
EOQ_i = \sqrt{\frac{2A_i D_i}{H_i}}
\]

- \(EOQ_1 = 50\) units
- \(EOQ_2 = 55\) units

**Budget constraint:** \(50(Q_1) + 80(Q_2) \leq 5000\)

\(50(50) + 80(55) = 6900 > 5000\)

*constraint is violated, not optimal*

---

Example A Continued

**Step 2:** Obtain \(Q^*_i\) and \(\lambda^*\) by applying method of Lagrange multiplier

\[
Q^*_i = \sqrt{\frac{2A_i D_i}{H_i + 2\lambda c_i}} \quad \text{for } i = 1, 2
\]

\[
Q^*_1 = \sqrt{\frac{2(50)(250)}{10 + 2\lambda(50)}} = \sqrt{\frac{25000}{10 + 100\lambda}} = \frac{50}{\sqrt{1 + 10\lambda}}
\]

\[
Q^*_2 = \sqrt{\frac{2(50)(484)}{16 + 2\lambda(80)}} = \sqrt{\frac{48400}{16 + 160\lambda}} = \frac{55}{\sqrt{1 + 10\lambda}}
\]
Example A Continued

**Step 3:** Substitute $Q^*_1$ and $Q^*_2$ into the budget constraint for $\lambda^*$:

$$Q^*_1 = \frac{50}{\sqrt{1 + 10\lambda}} \quad Q^*_2 = \frac{55}{\sqrt{1 + 10\lambda}}$$

$$50 \frac{50}{\sqrt{1 + 10\lambda}} + 80 \frac{55}{\sqrt{1 + 10\lambda}} = 5000$$

$$\frac{6900}{\sqrt{1 + 10\lambda}} = 5000 \quad \Rightarrow \quad \sqrt{1 + 10\lambda} = 1.38$$

---

Example A Continued

**Step 4:** Determine $Q^*_i$ values using $\lambda^*$:

$$Q^*_1 = \frac{50}{\sqrt{1 + 10\lambda}} \quad Q^*_2 = \frac{55}{\sqrt{1 + 10\lambda}}$$

$$\sqrt{1 + 10\lambda} = 1.38$$

$$Q^*_1 = \frac{50}{1.38} = 36 \text{ units}, \quad Q^*_2 = \frac{55}{1.38} = 40 \text{ units}$$


Multiple Items EOQ With Two Constraints

- Two common constraints in inventory systems are space and budget
- When both are involved in the same system, we will extend the procedure to a two-constraint case

Problem Formulations

minimize $TC(Q) = \sum_{i=1}^{m} TC_i(Q) = \sum_{i=1}^{m} \left( c_i D + H_i \frac{Q_i}{2} + \frac{A_i D_i}{Q_i} \right)$

\[ \sum_{i=1}^{m} c_i Q_i \leq C \quad \text{(Budget constraint)} \]

\[ \sum_{i=1}^{m} f_i Q_i \leq F \quad \text{(Space constraint)} \]

$Q_i \geq 0 \quad i = 1, 2, \ldots, m$
Solution Procedure

1. Solve the unconstraint problem. If both constraints are satisfied, this solution is the optimal one.
2. Otherwise include one of the constraints, say budget, and solve a one–constraint problem to find $Q_i$. If the space constraint is satisfied, this solution is the optimal one.
3. Otherwise, repeat the process for only the space constraint. If both single-constraint solutions do not yield the optimal solution, then both constraints are active, and the Lagrangian equation with both constraints must be solved:

$$TC(Q, \lambda_1, \lambda_2) = \sum_{i=1}^{m} TC_i(Q) + \lambda_1 \left[ \sum_{i=1}^{m} c_iQ_i - C \right] + \lambda_2 \left[ \sum_{i=1}^{m} f_iQ_i - F \right]$$

Example B

- Consider example A. Company has a total of 2000 units of space to store disk drives.
- disk drive 1 requires 25 units of space,
- disk drive 2 requires 40 units of space.
Formulations

\[ \min TC(Q) = \sum_{i=1}^{n} TC_i(Q) = 50(250) + 10 \frac{Q_1}{2} + \frac{50(250)}{Q_1} + 80(484) + 16 \frac{Q_2}{2} + \frac{50(484)}{Q_2} \]

Subject to:

\[ 50(Q_1) + 80(Q_2) \leq 5000 \quad \text{budget constraint} \]

\[ 25(Q_1) + 40(Q_2) \leq 2000 \quad \text{space constraint} \]

\[ Q_1, Q_2 \geq 0 \]

Example B Continued

**Step 1:** solve the unconstrained problem

\[ EOQ_i = \sqrt{\frac{2A_iD_i}{H_i}} \]

\[ EOQ_1 = \sqrt{\frac{2(50)(250)}{10}} = 50 \text{ units/order} \]

\[ EOQ_2 = \sqrt{\frac{2(50)(484)}{16}} = 55 \text{ units/order} \]

(1) Budget constraint: \( 50(50) + 80(55) = 6900 > 5000 \) not satisfied!

(2) Space constraint: \( 25(50) + 40(55) = 3450 > 2000 \) not satisfied!
Step 2: solve the problem only with budget constraint:

\[ Q^*_1 = \frac{50}{\sqrt{1 + 10\lambda}} = \frac{50}{1.38} = 36 \text{ units/order} \]

\[ Q^*_2 = \frac{55}{\sqrt{1 + 10\lambda}} = \frac{55}{1.38} = 40 \text{ units/order} \]

Check the solution to see if it satisfies the space constraint:

\[ 25(36) + 40(40) = 2500 > 2000 \text{ not satisfied!} \]

Step 3: solve the problem only with space constraint:

\[ Q^*_{i} = \sqrt{\frac{2AD_i}{H_i + 2\lambda f_i}} \quad \text{for } i = 1, 2 \]

\[ Q^*_1 = \sqrt{\frac{2(50)(250)}{10 + 2\lambda(25)}} = \frac{50}{\sqrt{1 + 5\lambda}} \]

\[ Q^*_2 = \sqrt{\frac{2(50)(484)}{16 + 2\lambda(40)}} = \frac{55}{\sqrt{1 + 5\lambda}} \]
Example B Continued

\[ 25 \frac{50}{\sqrt{1+5\lambda}} + 40 \frac{55}{\sqrt{1+5\lambda}} = 2000 \rightarrow \sqrt{1+5\lambda} = 1.73 \]

\[ Q^*_1 = \frac{50}{\sqrt{1+5\lambda}} = \frac{50}{1.73} = 28 \text{ units/order} \]

\[ Q^*_2 = \frac{55}{\sqrt{1+5\lambda}} = \frac{55}{1.73} = 32 \text{ units/order} \]

(1) Budget constraint: \( 50(28) + 80(32) = 3960 < 5000 \) satisfied!

(2) Space constraint: \( 25(28) + 40(32) = 1980 < 2000 \) satisfied!

Optimal Solution

<table>
<thead>
<tr>
<th>( Q_1 )</th>
<th>( Q_2 )</th>
<th>TC(Q)</th>
<th>Budget Constraint</th>
<th>Space Constraint</th>
<th>Increase in TC(Q) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>55</td>
<td>52600</td>
<td>infeasible</td>
<td>infeasible</td>
<td>0</td>
</tr>
<tr>
<td>36</td>
<td>40</td>
<td>52672</td>
<td>feasible</td>
<td>infeasible</td>
<td>0.14</td>
</tr>
<tr>
<td>28</td>
<td>32</td>
<td>52819</td>
<td>feasible</td>
<td>feasible</td>
<td>0.42</td>
</tr>
</tbody>
</table>